Clique-Helly graphs, and Hereditary Clique-Helly graphs — a mini-survey.

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Abstract

We describe clique-Helly and hereditary clique-Helly graphs, and give structural characterisations leading to polynomial-time recognition algorithms. We also relate these graphs to other graph classes defined by intersection conditions (especially conditions on cliques, or Helly-type conditions).

1 Introduction and definitions

All graphs we consider are finite, loopless and without multiple edges. A *clique* of a graph G is a maximal complete subgraph[1] of G. Every clique is uniquely specified by its vertices, and we define the size of a clique to be the number of vertices it contains. Similarly, we take the intersection of a family $\mathcal{F}$ of cliques to be the set of vertices common to all cliques in $\mathcal{F}$; and the size of the intersection to be the number of vertices that are common to all cliques of $\mathcal{F}$. A clique is trivial if it has size 1. We will use n and m for the number of vertices and edges of a graph G, and $\omega(G)$ for the maximum clique size.

[1]Caution: many authors use clique to mean "complete subgraph", and maximal clique for "maximal complete subgraph".

A set family $\mathcal{F}$ has the *Helly property* if every collection of pairwise-intersecting sets from $\mathcal{F}$ has a common point. In other words, if $\{S_1, \ldots, S_r\} \subset \mathcal{F}$, then

$$(\forall i, j, S_i \cap S_j \neq \emptyset) \Rightarrow (\bigcap_{k=1}^{r} S_k \neq \emptyset). \quad \ddagger$$

A graph G is *clique-Helly* if the cliques of G have the Helly property. G is *hereditary clique-Helly* if every induced-subgraph of G is clique-Helly. We will sometimes abbreviate "clique-Helly" to CH, and "hereditary clique-Helly" to HCH; on occasion we might use CH or HCH to denote the set of all CH graphs, or all HCH graphs. The smallest graph that is *not* clique-Helly is the Hajós graph (Fig. 1).

We will first characterise these two classes of graphs, giving polynomial-time recognition algorithms in the process; and then (mostly without proofs) relate them to other classes. Containments between the various classes are shown in Figure 3, where an arrow from A to B means that A properly contains B. Some of the classes mentioned briefly in the last section are not included in the diagram; more importantly, there might be some missing arrows — we have only included those that we know of from the results in this paper; in other words, the lack of a directed path from A to B need not mean that the two classes are incomparable. Dragan's dissertation [9] reportedly includes many relevant results, but we did not have access to it, so some attributions might be missing.

2 Clique-Helly graphs

It is easy to construct clique-Helly graphs [14] — for any graph H, the graph $H + K_1$ (formed by adding a vertex v that is adjacent to all of $V(H)$) is clique-Helly as every clique contains v. So every graph is an induced-subgraph of a clique-Helly graph with just one more vertex. In particular, the k-partite graph H_k with all parts of size 3 has $3k$ vertices but 3^k cliques, and is not clique-Helly when $k \geq 3$; on the other hand, $H_k + K_1$ is a clique-Helly graph with $3k + 1$ vertices and 3^k cliques.

In recognising clique-Helly graphs, a brute-force verification of the Helly property would be clearly prohibitive, especially since we have just seen that there are both CH graphs and non-CH graphs with exponentially many cliques. Roberts and Spencer discovered a shortcut for testing the Helly property. We will not use it here, but it is worth mentioning. A family of sets is *m-Helly* if ‡ is satisfied for collections $\{S_1, \ldots, S_r\} \subset \mathcal{F}$ with $r \leq m$.

1. Theorem [20]. *Let $\omega(G) \leq m$, and let $\mathcal{F}$ be a collection of complete subgraphs of G. Then $\mathcal{F}$ is Helly if and only if it is m-Helly.* $\square$

The test that we will use was discovered by Claude Berge, who reported it to Roberts and Spencer [20]. For a set family $\mathcal{F} = \{S_1, \ldots, S_p\}$, we use $\cup\mathcal{F}$ to denote $\cup_{i=1}^{p} S_i$, and $\cap\mathcal{F}$ to denote $\cap_{i=1}^{p} S_i$. For any three (not necessarily distinct) points $x, y, z \in \cup\mathcal{F}$, we define $\mathcal{F}(x, y, z)$ to be the subfamily of sets from $\mathcal{F}$ that contain at least two of x, y, z.

2. Theorem. *A set family $\mathcal{F}$ has the Helly property iff $\forall x, y, z \in \cup\mathcal{F}$:*

$$\mathcal{F}(x, y, z) \neq \emptyset \Rightarrow \cap\mathcal{F}(x, y, z) \neq \emptyset.$$

Proof: It is clear that any two sets in $\mathcal{F}(x, y, z)$ must intersect in either x, y or z; thus $\cap\mathcal{F}(x, y, z) \neq \emptyset$ by the Helly property.

Conversely, let $\mathcal{F}$ satisfy the stated condition. We want to show that for any subfamily $\mathcal{F}' = \{S'_1, \ldots, S'_r\}$ of $\mathcal{F}$ such that $S'_i \cap S'_j \neq \emptyset$ whenever $1 \leq i \leq j \leq r$, we have $\cap\mathcal{F}' \neq \emptyset$. If r is 1 or 2 this is trivial. Otherwise, for $i = 1, 2, 3$ define $\mathcal{F}'_i := \mathcal{F}' \setminus S'_i$. By induction on r, $\cap\mathcal{F}'_i \neq \emptyset$, so we can find $x_i \in \cap\mathcal{F}'_i$. Now every S'_j is in at least two of $\mathcal{F}'_1, \mathcal{F}'_2, \mathcal{F}'_3$, therefore it contains at least two (not necessarily distinct) vertices from $\{x_1, x_2, x_3\}$. Then $\mathcal{F}'(x_1, x_2, x_3) \supseteq \mathcal{F} \neq \emptyset$, so $\cap\mathcal{F}'(x_1, x_2, x_3) \neq \emptyset$ and it follows that $\cap\mathcal{F}' \neq \emptyset$. $\square$

When $\mathcal{F}$ is a large family, computing $\mathcal{F}(x, y, z)$ and $\cap\mathcal{F}(x, y, z)$ is still a slow process. However, with another good idea we can use Theorem 2 to give an $O(nm^2)$ recognition algorithm for clique-Helly graphs. The first such algorithm was given by Dragan [9], but the exposition we give below is adapted from Szwarcfiter [21].

If $\mathcal{C}$ is a collection of subgraphs of G, then $\cup\mathcal{C} = (V', E')$ will denote the union of all the members of $\mathcal{C}$; that is, $V' := \cup_{H\in\mathcal{C}} V(H)$, and $E' := \cup_{H\in\mathcal{C}} E(H)$. Note that this need not be an induced subgraph of G. A *universal vertex* in a subgraph H of G is a vertex $v \in V(H)$ that is adjacent in H to all the other vertices of H.

Let T be a triangle of G. The *extended triangle* of T in G is the subgraph T_{ext} induced by the vertices which form a triangle with at least one edge of T. If T has vertices x, y, z, and $\mathcal{F}$ is the family of cliques of G, then $\mathcal{F}(x, y, z)$ is the collection of cliques that contain at least one edge of T; we define $T_{\text{clique}} := \cup\mathcal{F}(x, y, z)$.

3. Lemma [21]. *Let T be a triangle in G.*
Then T_{clique} is a spanning subgraph of T_{ext}.
Moreover, v is a universal vertex in T_{clique} iff it is universal in T_{ext}.
Proof: Any vertex $v \in V(T_{\text{clique}})$ is in a clique that contains some edge e of T; then v forms a triangle with e, so $v \in V(T_{\text{ext}})$. Conversely, if $v \in V(T_{\text{ext}})$, then v is in a triangle containing some edge e of T, and is thus in a clique containing e. Thus $V(T_{\text{clique}}) = V(T_{\text{ext}})$, and since T_{ext} is an induced subgraph of G we have $E(T_{\text{clique}}) \subseteq E(T_{\text{ext}})$.

Therefore if v is a universal vertex in T_{clique}, it is also universal in T_{ext}. Conversely, if v is a universal vertex in T_{ext}, and C is a clique in $\mathcal{F}(x, y, z)$, then v is equal or adjacent to every vertex of C, so v must be in C, by maximality of C; since this is true for every clique in $\mathcal{F}(x, y, z)$, v is a universal vertex in T_{clique}. □

4. Theorem [21]. *G is a clique-Helly graph if and only if all of the extended triangles in G have a universal vertex.*
Proof: We will show that the condition in Theorem 2 (with $\mathcal{F}$ being the set of cliques of G) is equivalent to the presence of a universal vertex in every extended triangle. Let x, y, z be vertices of G; then $\mathcal{F}(x, y, z)$ is the set of cliques that contain at least two of these vertices. If $\mathcal{F}(x, y, z)$ is not empty then at least two of the vertices must be adjacent. If they are not all adjacent then, say, x is not adjacent to y; in this case, any clique in $\mathcal{F}(x, y, z)$ must contain z, so $\cap\mathcal{F}(x, y, z) \neq \emptyset$.

Otherwise, x, y, z form a triangle T. Now a vertex v is in $\cap\mathcal{F}(x, y, z)$ iff it is in all the cliques in $\mathcal{F}(x, y, z)$, iff it is adjacent to all the vertices in T_{clique}. But this means that v is a universal vertex in T_{clique}, which by Lemma 3 is equivalent to being a universal vertex in T_{ext}. □

The algorithm now follows easily:

Algorithm Clique-Helly
for every triangle T of G
  compute T_{ext};
  if T_{ext} has no universal vertex
    stop and output "G is not clique-Helly."
Output "G is clique-Helly."

The number t of triangles of G is $O(nm)$, and they can be all found in time $O(nm)$. For each one, T_{ext} can be found in time $O(n)$, and we can check for a universal vertex in time $O(m)$. Thus the algorithm runs in time $O(nm + t(n+m)) = O(nm^2)$.

Another interesting characterisation is given by Bandelt, Farber and Hell [3]. The *open neighbourhood* of v, denoted $N(v)$, is the set of vertices adjacent to v; the closed neighbourhood is defined by $N[v] := N(v) \cup \{v\}$. G is open-neighbourhood-Helly if the collection of open neighbourhoods satisfies the Helly property. For a graph G we define a bipartite graph $B(G)$ as follows: $V(B(G)) := \{v' \mid v \in V(G)\} \cup \{v'' \mid v \in V(G)\}$; $E(B(G)) := \{(v', w'') \mid v \in N[w] \text{ or } w \in N[v]\}$.

5. Theorem [3]. *G is clique-Helly iff $B(G)$ is open-neighbourhood-Helly.* □

3 Hereditary clique-Helly graphs

G is hereditary clique-Helly (HCH) iff all its induced-subgraphs are clique-Helly. There are several characterisations of HCH graphs, some involving the forbidden graphs shown in Figure 1, also known as *ocular graphs* [23]; we label their vertices as shown on the first graph. G_1 is particularly noteworthy, as it is contained in all the others and is thus the smallest graph that is not clique-Helly; it is also known as the Hajós graph, and the 3-sun (S_3). We will say that G is *Hajós-free* (or *ocular-free*) if it has no induced subgraph isomorphic to G_1 (or G_1, G_2, G_3, G_4).

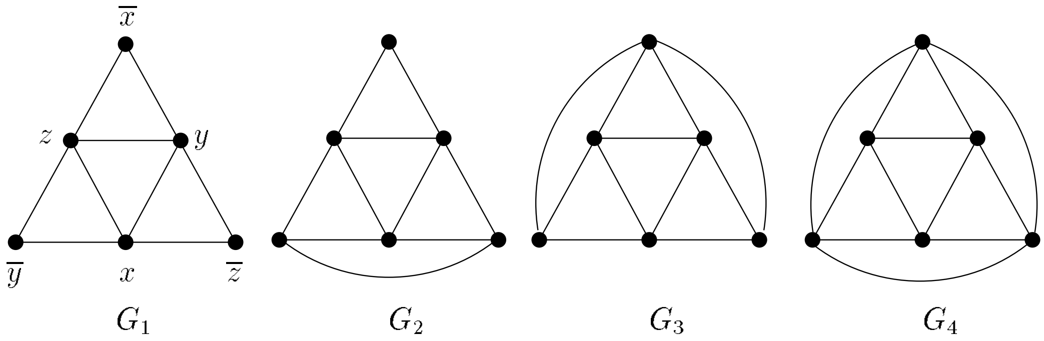


Figure 1: The ocular graphs; G_1 is the Hajós graph.

6. Theorem [19]. *G is hereditary clique-Helly if and only if it is ocular-free.*

Proof: None of the ocular graphs are clique-Helly (take the center triangle T of G_i, and note that $T_{\rm ext} = G_i$ has no universal vertex); thus G cannot contain any G_i.

Conversely, if G is not HCH, it contains some induced-subgraph F that is not clique-Helly. By Theorem 4, F contains a triangle T for which $T_{\rm ext}$ has no universal vertex. Let the vertices of T be $\{x, y, z\}$. Since x is not universal in $T_{\rm ext}$, there is a vertex $\overline{x}$ that is not adjacent to x, but is adjacent to y and z (since $\overline{x}$ is in $T_{\rm ext}$). Similarly, there is a vertex $\overline{y}$ adjacent to x and z but not y, and a vertex $\overline{z}$ adjacent to x and y but not z. Then $\{x, y, z, \overline{x}, \overline{y}, \overline{z}\}$ induce one of the graphs of Figure 1. □

This immediately gives an $O(n^6)$ recognition algorithm, but we can do better by providing a uniform description of the four forbidden graphs of Figure 1.

An edge uv of a triangle uvw is *good* if any vertex adjacent to both u and v is also adjacent to w. The center triangle in each of the forbidden subgraphs has no good edge, and whenever we have such a triangle we must have one of the forbidden graphs. We thus have:

7. Theorem [19]. *A graph is HCH if and only if every triangle has a good edge.* □

This gives an $O(n^2m)$ recognition algorithm (since there are $O(nm)$ triangles):

ALGORITHM HEREDITARY CLIQUE-HELLY
FOR every triangle $T = \{x, y, z\}$ of G:
  $M := E(T)$; (the candidates for good edges)
  FOR every $v \notin \{x, y, z\}$
    FOR $e \in M$
      IF $N(v) \cap \{x, y, z\} = V(e)$, THEN $M := M \setminus e$; (e is not good)
    IF $M = \emptyset$, THEN STOP and output "G is not HCH."
Output "G is HCH."

We strengthened the definition of clique-Helly graphs to obtain hereditary clique-Helly graphs. There is another restriction that we could have imposed instead: G is a *strong clique-Helly graph* (or SCH) if for any collection $\{C_1, \ldots, C_r\}$ of cliques of G:

$$|\bigcap_{k=1}^{r} C_k| = \min_{1 \le i < j \le r} |C_i \cap C_j|. \quad (*)$$

If this relationship holds whenever the C_i's are complete subgraphs (not necessarily maximal), then G is *strong complete-Helly.* It is easy to see that G is strong complete-Helly iff it is triangle-free.

8. Theorem [23]. *For any graph G the following are equivalent:*

(i). G is strong clique-Helly

(ii). for any three distinct cliques C_1, C_2, C_3 of G,

$$|C_1 \cap C_2 \cap C_3| = \min\{|C_1 \cap C_2|, |C_2 \cap C_3|, |C_3 \cap C_1|\} \quad (**)$$

(iii). G is ocular-free.

Proof: (i $\Leftrightarrow$ ii). Clearly (i) $\Rightarrow$ (ii). Conversely, if (ii) holds, then given cliques $\{C_1, \ldots, C_r\}$, where wlog $|C_1 \cap C_2 \cap C_3| = |C_2 \cap C_3|$, we have

$$|\bigcap_{k=1}^{r} C_k| = |(C_1 \cap C_2 \cap C_3) \cap \bigcap_{k=4}^{r} C_k| = |(C_2 \cap C_3) \cap \bigcap_{k=4}^{r} C_k| = |\bigcap_{k=2}^{r} C_k|,$$

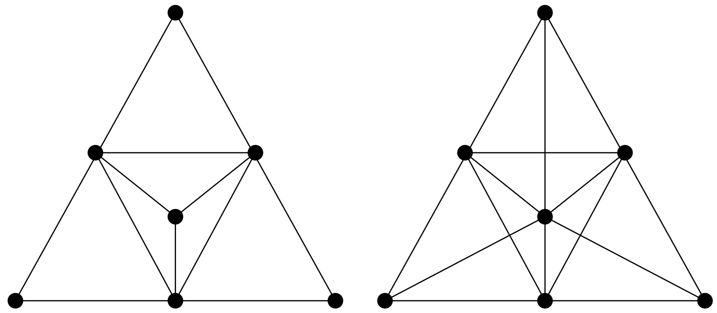
and by applying this argument another $r-3$ times we see that (*) is satisfied.
(ii $\Rightarrow$ iii). Now suppose (ii) does not hold. Then $|C_1 \cap C_2| > |C_1 \cap C_2 \cap C_3|$, so we can find a vertex $x \in (C_1 \cap C_2) \setminus C_3$, and similarly we have vertices $y \in (C_2 \cap C_3) \setminus C_1$ and $z \in (C_3 \cap C_1) \setminus C_2$. Since C_3 is a *maximal* complete subgraph, and $x \notin C_3$, there is a vertex $\overline{x} \in C_3$ that is not adjacent to x; similarly, there is a vertex $\overline{y} \in C_1$ that is not adjacent to y, and a vertex $\overline{z} \in C_2$ that is not adjacent to z. Then $\{x, y, z, \overline{x}, \overline{y}, \overline{z}\}$ induce one of the graphs of Figure 1.
(iii $\Rightarrow$ ii). Let G contain one of $G_1, \ldots, G_4$. Then the complete subgraphs $C_1' := \{x, y, \overline{z}\}$, $C_2' := \{y, z, \overline{x}\}$ and $C_3' := \{z, x, \overline{y}\}$ do not satisfy (**). We now extend them arbitrarily to maximal cliques, say C_1, C_2, C_3. Note that $\overline{x}$ cannot be in any clique with x, $\overline{y}$ cannot be in any clique with y, and z is not in any clique that contains $\overline{z}$. Thus $C_1 = C_1' \cup E_1$, where E_1 does not contain any of $\{x, y, z, \overline{x}, \overline{y}, \overline{z}\}$. Similarly $C_2 = C_2' \cup E_2$ and $C_3 = C_3' \cup E_3$. Then

$$\begin{aligned}
& |C_1 \cap C_2 \cap C_3| = |C_1' \cap C_2' \cap C_3'| + |E_1 \cap E_2 \cap E_3| = |E_1 \cap E_2 \cap E_3| \\
\leq\ & \min\{|E_1 \cap E_2|, |E_2 \cap E_3|, |E_3 \cap E_1|\} \\
<\ & \min\{1 + |E_1 \cap E_2|, 1 + |E_2 \cap E_3|, 1 + |E_3 \cap E_1|\} \\
=\ & \min\{|C_1' \cap C_2'| + |E_1 \cap E_2|, |C_2' \cap C_3'| + |E_2 \cap E_3|, |C_3' \cap C_1'| + |E_3 \cap E_1|\} \\
=\ & \min\{|C_1 \cap C_2|, |C_2 \cap C_3|, |C_3 \cap C_1|\}. \qquad \square
\end{aligned}$$

Note that this implies that every induced subgraph of a strong clique-Helly graph is also strong clique-Helly, a fact that is not obvious from the definition.

A further restriction of strong clique-Helly graphs is the following: G is *semi-Ptolemaic* if, for any collection $\mathcal{C}$ of pairwise intersecting cliques of G, for every $C_i \in \mathcal{C}$ there is a $C_j \in \mathcal{C}$ such that $C_i \cap C_j = \cap\mathcal{C}$. A connected graph G is *Ptolemaic* provided that for any four vertices u_i, $1 \leq i \leq 4$, of G, the six distances $d_{ij} = d_G(u_i, u_j)$, $i \neq j$, satisfy the Ptolemaic inequality $d_{12}d_{34} \leq d_{13}d_{24} + d_{14}d_{23}$. Howorka [16] showed that every Ptolemaic graph is semi-Ptolemaic; in fact, G is Ptolemaic iff it is semi-Ptolemaic and chordal.

A graph G is *clique-irreducible* if every (non-trivial) clique contains an edge that is not in any other clique; G is *hereditary clique-irreducible* (or HCI) if every induced subgraph of G is clique-irreducible. The classes of clique-Helly graphs and clique-irreducible graphs are incomparable: the graph F_1



F_1 F_2

Figure 2: A clique-irreducible and a clique-Helly graph.

in Figure 2 is clique-irreducible but not clique-Helly, while F_2 is clique-Helly but not clique-irreducible. However, the hereditary clique-irreducible graphs are precisely the hereditary clique-Helly graphs:

9. Theorem [23]**.** *G is hereditary clique-irreducible iff it is ocular-free.*
Proof: None of the graphs in Figure 1 are clique-irreducible (the edges of the center triangle are all contained in other cliques); thus if G is HCI, it cannot contain any G_i.

Conversely, if G is not hereditary clique-irreducible, it contains some induced-subgraph F that is not clique-irreducible. We can find a minimum number r of cliques $C_1, \ldots, C_r$ of F which together cover another clique, C. By minimality of r, $C \cap C_1 \nsubseteq C_2 \cup \cdots \cup C_r$; so there is an edge $xy \in C \cap C_1$ that is not in any of $C_2, \ldots, C_r$. There must also be a vertex z in $C \setminus C_1$; then yz is not in C_1, so wlog it is in C_2, and thus $x \notin C_2$ since $xy \notin C_2$. Therefore zx cannot be in C_1 or C_2 (since $z \notin C_1$ and $x \notin C_2$), so wlog it is in C_3.

Now we can find a vertex $\overline{x}$ in C_2 that is not adjacent to x; similarly, there is a vertex $\overline{y}$ in C_3 that is not adjacent to y, and a vertex $\overline{z}$ in C_1 that is not adjacent to z. Then $\{x, y, z, \overline{x}, \overline{y}, \overline{z}\}$ induce one of the graphs of Figure 1. $\square$

The forbidden subgraphs mentioned in this theorem all contain triangles, so triangle-free graphs are trivially hereditary clique-irreducible. For graphs with maximum clique-size at most 3 we have the following result (clique-graphs will be defined later on), where the last two statements are easily seen to be equivalent by Theorem 9:

10. Theorem [20, 23]. *Let G be a graph with $\omega(G) \leq 3$. Then the following are equivalent:*

(i). G is a clique-graph

(ii). G is clique-irreducible

(iii). G is hereditary clique-irreducible $\square$

We have seen (at the beginning of Section 2) that there are CH graphs with exponentially many cliques. On the other hand, the number of cliques in an irreducible graph is bounded by the number of edges; this bound holds in particular for the subclass of HCH graphs. Tsukiyama et al. give an algorithm that generates all the cliques of a graph in time $O(nmc)$, where c is the number of cliques, and we thus have:

11. Theorem [19, 22]. *The number of cliques of an irreducible (and, in particular, HCH) graph is $O(m)$. The max-clique problem can thus be solved in time $O(nm^2)$ for these graphs.* $\square$

4 Strictly clique-irreducible graphs

We saw before that the definition of clique-Helly graphs could be strengthened in two ways, giving us hereditary CH-graphs and strong CH-graphs, which actually turned out to be the same. We have already restricted clique-irreducible graphs to hereditary clique-irreducible graphs (which also turned out to be equivalent to hereditary CH-graphs); we now define *strictly clique-irreducible* graphs to be the graphs without multicliqual edges, that is, graphs where each edge lies in a unique clique. It is not difficult to see that they are precisely the graphs without $K_4 - e$ as an induced subgraph. Since the forbidden subgraphs in Figure 1 contain $K_4 - e$, the strictly clique-irreducible graphs are a proper subclass of the hereditary clique-irreducible graphs.

There are a number of different characterisations of this class, but first we need some definitions. A family $\mathcal{C}$ of complete subgraphs is a *complete covering* of G if every edge of G lies in at least one member of $\mathcal{C}$; the *minimum* *complete covering number* $mcc(G)$ is the minimum cardinality of a complete covering, and a *minimum complete covering* is a complete covering $\mathcal{C}$ with $|\mathcal{C}| = mcc(G)$. A family $\mathcal{C}$ of cliques is a *clique partition* if every edge lies in exactly one member of $\mathcal{C}$; the definitions of a *minimum clique partition* and the *minimum clique partition number* $mcp(G)$ are analogous to the ones for complete covers. The number of cliques of G is denoted by $c(G)$.

Not every graph has a clique partition, so $mcp(G)$ may not be defined, but if it is then $mcc(G) \leq mcp(G) \leq c(G)$, with equality, for example, when G is triangle-free. It is also easy to see that a graph is clique-irreducible if and only if $mcc(G) = c(G)$. We can now state Wallis and Zhang's characterisations of strictly clique-irreducible graphs:

12. Theorem [23]. *For any graph G the following are equivalent:*

(i). *G is strictly clique-irreducible (i.e., each edge of G is in a unique clique)*

(ii). *G has no induced subgraph isomorphic to $K_4 - e$*

(iii). $mcp(G) = c(G)$

(iv). *the set of cliques is the unique minimum clique partition of G*

(v). *G is clique-irreducible and $mcp(G)$ is defined (i.e., each clique has an edge that is not contained in any other clique, and there is a clique partition of G)*

(vi). *G has a collection of complete subgraphs that satisfies the Helly property and also partitions the edge-set of G.* □

5 Clique graphs of clique-Helly graphs, and other classes

The clique graph of G, denoted $K(G)$, is the intersection graph defined by the cliques of G. That is, $V(K(G))$ is the set of cliques of G, and $(C_i, C_j) \in E(K(G)) \Leftrightarrow C_i \cap C_j \neq \emptyset$.

Hamelink [14] was the first to show that every clique-Helly graph is the clique graph of some graph; he also showed that no graph that contained the Hajós graph (see Fig. 1) as an induced-subgraph could be a clique graph. Roberts and Spencer [20] generalised this by characterising clique graphs as those graphs whose edges can be covered by a family $\mathcal{F}$ of complete subgraphs (not necessarily maximal) such that $\mathcal{F}$ satisfies the Helly property; when $\mathcal{F}$ is the set of maximal cliques of G we get Hamelink's result.

The classes of clique-graphs and clique-irreducible graphs are incomparable: the graph F_1 in Figure 2 is clique-irreducible but is not a clique-graph (since it contains the Hajós graph), while F_2 is a clique-graph (since it is clique-Helly) but not clique-irreducible.

The r^{th} *iterated clique graph* of G, denoted $K^r(G)$, is $K(K^{r-1}(G))$, where $K^1(G) := K(G)$. G is *clique-periodic* with period p if the sequence $\{G, K(G), K^2(G), K^3(G), \ldots\}$ of iterated clique-graphs of G eventually repeats itself; more precisely, p is the smallest natural number such that:

$$\exists N \mid \forall r \geq N, K^r(G) = K^{r+p}(G).$$

If the period is 1, we say that G is *clique-convergent* to $K^N(G)$.

A *disk* (with center v and radius r, for some $v \in V(G)$ and $r \in \mathbb{N}$) is the set of vertices of distance at most r from v. The *closed neighbourhood* of v is the disk with center v and radius 1, denoted $N[v]$. G is *disk-Helly* if the set of disks (with arbitrary center and radius) has the Helly property, and *neighbourhood-Helly* if the set of closed neighbourhoods has the Helly property. Every clique is clearly contained in the closed neighbourhood of any of its vertices. Thus, every disk-Helly graph is neighbourhood-Helly, and every neighbourhood-Helly graph is clique-Helly.

The best known algorithms for recognising disk-Helly, neighbourhood-Helly and clique-Helly graphs run in time $O(n^2m)$ [4, 9], $O(n^2m)$ [9] and $O(nm^2)$ [9, 21], respectively. Disk-Helly graphs were characterised by Dragan [10], and also by Bandelt and Prisner:

13. Theorem [2]**.** *The clique-Helly graphs that are clique-convergent to the one-vertex graph (that is, $K^r(G) = K_1$ for some r) are precisely the disk-Helly graphs. Moreover, the clique graphs of chordal graphs are disk-Helly (and thus chordal graphs are also clique-convergent to K_1).* □

Clique-Helly graphs that are not clique-convergent are still clique-periodic:

14. Theorem [12]. *If G is clique-Helly, then $K^2(G)$ is (isomorphic to) an induced-subgraph of G. Thus clique-Helly graphs are clique-periodic with period at most 2.*

Proof: We will sketch the proof. Define an equivalence relationship where $v \simeq w$ iff $N[v] = N[w]$; note that each equivalence class is a complete subgraph, and every clique (and thus every intersection of cliques) is some union of equivalence classes. Take a representative from each class to get a maximal induced subgraph G' where no two vertices have the same closed neighbourhood. Then $K(G') = K(G)$, and G' is still clique-Helly.

We will say that a vertex y dominates z if $N[y] \supseteq N[z]$. No two vertices of G' dominate each other. We claim that the the subgraph G'' induced by the undominated vertices in G' is isomorphic to $K^2(G') = K^2(G)$.

We will first show that the cliques of $K(G')$ are in one-to-one correspondence with the undominated vertices of G'. A clique of $K(G')$ corresponds to a maximal set of pairwise intersecting cliques of G', say $S = \{C_1, \ldots, C_r\}$; because G' is clique-Helly, these cliques share a common vertex v. Because S is maximal, $N[v]$ is exactly $C_1 \cup \cdots \cup C_r$; and because no other vertex has the same closed neighbourhood, S uniquely determines v. Using the maximality of S again, v cannot be dominated by any other vertex; conversely, any undominated vertex is the unique intersection of all the cliques that contain it. Thus $V(G'') = V(K^2(G))$.

Finally, whenever two vertices v and w of G'' are adjacent, the corresponding sets of cliques intersect in a clique that contains vw. And if two sets S_1, S_2, of pairwise-intersecting cliques share a common clique C, then C contains the common intersection of S_1 and the common intersection of S_2. $\square$

Wallis and Zhang [23] proved this result for the proper subclass of strictly clique-irreducible graphs, showing in that case that if V_2 is the set of vertices contained in at least two cliques of G, then $K^2(G) \cong G[V_2]$.

We end our survey with an assortment of definitions and results about some proper subclasses of clique-Helly graphs. Let C be a cycle with vertices labeled consecutively $v_1, \ldots, v_k$; the distance between two vertices along the cycle is $d_C(v_i, v_j) := \min\{|i-j|, k-|i-j|\}$. An edge (v_i, v_j) is a *chord* if $d_C(v_i, v_j) > 1$; the chord is *odd* if, moreover, $d_C(v_i, v_j)$ is odd. Chords and odd chords can only appear in cycles of length at least 4 or 6, respectively.

G is *chordal* if every cycle of length at least 4 has a chord; if, moreover, every cycle of length at least 6 has an odd chord, then G is *strongly chordal*.

An *interval graph* is an intersection graph of intervals on a line; if the intervals all have the same length, we have a *unit interval graph* (also known as time graphs or indifference graphs). Every interval graph is known to be strongly chordal.

For any class $\mathcal{C}$ of graphs we define $K(\mathcal{C}) := \{K(G) \mid G \in \mathcal{C}\}$. If $K(\mathcal{C}) = \mathcal{C}$, we say that $\mathcal{C}$ is *fixed under* K. In other words, G is in $\mathcal{C}$ if and only if it is the clique graph of *some* graph H in $\mathcal{C}$; note, however, that it might well also be the clique graph of some graph J that is *not* in $\mathcal{C}$. For example, $K(\{K_1\}) = \{K_1\}$, but $K(K_n)$ is also K_1 for any n.

Not very many classes are known to be fixed under K, but Bandelt and Prisner [2] described a whole family of graph classes that are fixed under K, in the process giving a unified proof (via hypergraphs) for six specific classes. The bibliography of their paper is also a good starting point for results on clique-graphs and iterated clique-graphs.

15. Theorem. *The following classes of graphs are fixed under K:*

(i). clique-Helly graphs [12, 2];

(ii). hereditary clique-Helly graphs [2, 19];

(iii). strictly clique-irreducible graphs [17, 23, 2];

(iv). disk-Helly graphs [2];

(v). strongly chordal graphs [2];

(vi). unit interval graphs [15]. □

A referee of Hedman's paper [15] noted that in fact the clique graphs of interval graphs are precisely the unit-interval graphs. Hedman also showed that unit interval graphs are clique-convergent to K_1, a special case of Theorem 13.

G is *Helly-chordal* if it is clique-Helly and chordal; *clique-chordal* if $K(G)$ is chordal; *dually-chordal* if it is clique-Helly and clique-chordal; and *doubly-chordal* if it is Helly-chordal and clique-chordal (equivalently, if it is chordal and dually-chordal). G is *hereditary dually-chordal* if every induced-subgraph is dually-chordal, and *hereditary disk-Helly* if every induced-subgraph is disk-Helly.

16. Theorem [6, 11]. *The clique graphs of chordal graphs are precisely the dually-chordal graphs, and the clique graphs of dually-chordal graphs are precisely the Helly-chordal graphs. Thus, by Thm. 13, every dually-chordal graph is disk-Helly, and so every hereditary dually-chordal graph is hereditary disk-Helly.* □

17. Theorem [9]. *Every Helly-chordal graph is disk-Helly.* □

18. Theorem [9]. *G is hereditary disk-Helly iff it is chordal and Hajós-free.*

19. Theorem [8, 13]. *G is strongly chordal iff it is hereditary dually-chordal.*

For any integer $k \geq 1$ the k^{th} power of $G = (V, E)$ is defined by $G^k := (V, E')$, where $E' := \{(v, w) \mid d(v, w) \leq k\}$ and $d(v, w)$ is the distance from v to w in G.

20. Theorem [9]. *If G is disk-Helly, dually-chordal or doubly-chordal, then for all $k \geq 1$, G^k is (respectively) disk-Helly, dually-chordal or doubly-chordal.* □

21. Theorem [1, 11]. *Let G be a bipartite graph. Then G is disk-Helly if and only if G^k is neighbourhood-Helly for all $k \geq 1$.* □

22. Theorem [6]. *A graph H is dually-chordal iff H is the clique-graph of some chordal graph. A graph H is Helly-chordal iff H is the clique-graph of some dually-chordal graph. In particular [2], if G is chordal, then $K^2(G)$ is always chordal; it can also be shown [11] that for G chordal, $K(G)$ is chordal iff G^2 is chordal.* □

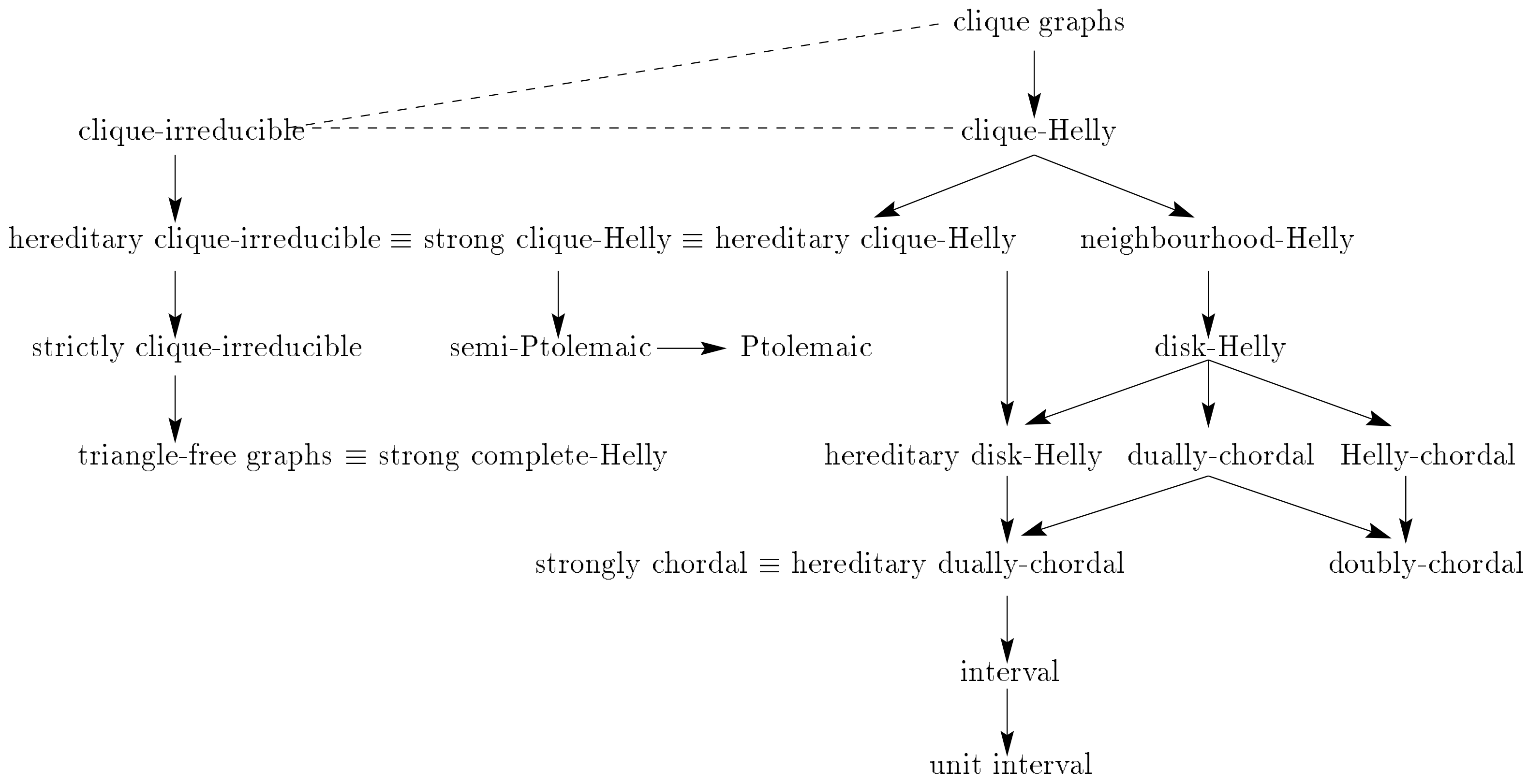


Figure 3: A (partial) class diagram: arrows denote proper containment, dashed lines denote incomparability.

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