New Results on Generalized Graph Coloring

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Abstract

For graph classes $\mathcal{P}_1, \ldots, \mathcal{P}_k$, Generalized Graph Coloring is the problem of deciding whether the vertex set of a given graph G can be partitioned into subsets $V_1, \ldots, V_k$ so that V_j induces a graph in the class $\mathcal{P}_j$ $(j = 1, 2, \ldots, k)$. If $\mathcal{P}_1 = \cdots = \mathcal{P}_k$ is the class of edgeless graphs, then this problem coincides with the standard vertex k-COLORABILITY, which is known to be NP-complete for any $k \geq 3$. Recently, this result has been generalized by showing that if all $\mathcal{P}_i$'s are additive induced-hereditary, then the generalized graph coloring is NP-hard, with the only exception of bipartite graphs. Clearly, a similar result follows when all the $\mathcal{P}_i$'s are co-additive.

In this paper, we study the problem where we have a mixture of additive and co-additive classes, presenting several new results dealing both with NP-hard and polynomial-time solvable instances of the problem.

Keywords: Generalized Graph Coloring; Polynomial algorithm; NP-completeness

1 Introduction

All graphs in this paper are finite, without loops and multiple edges. For a graph G we denote by $V(G)$ and $E(G)$ the vertex set and the edge set of G, respectively. By $N(v)$ we denote the neighborhood of a vertex $v \in V(G)$, i.e. the subset of vertices of G adjacent to v. The subgraph of G induced by a set $U \subseteq V(G)$ will be denoted

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$G[U]$. We say that a graph G is H-free if G does not contain H as an induced subgraph. As usual, K_n and P_n stand for the complete graph and chordless path on n vertices, respectively, and the complement of a graph G is denoted $\overline{G}$.

A class of graphs, or synonymously graph property, $\mathcal{P}$ is said to be *hereditary* if $G \in \mathcal{P}$ implies $G - v \in \mathcal{P}$ for any vertex $v \in V(G)$. We call $\mathcal{P}$ *monotone* if $G \in \mathcal{P}$ implies $G - v \in \mathcal{P}$ for any vertex $v \in V(G)$ and $G - e \in \mathcal{P}$ for any edge $e \in E(G)$. Clearly every monotone property is hereditary, but the converse statement is not true in general. A property $\mathcal{P}$ is *additive* if $G_1 \in \mathcal{P}$ and $G_2 \in \mathcal{P}$ with $V(G_1) \cap V(G_2) = \emptyset$ implies $G = (V(G_1) \cup V(G_2), E(G_1) \cup E(G_2)) \in \mathcal{P}$. The class of graphs containing no induced subgraphs isomorphic to graphs in a set Y will be denoted $Free(Y)$. It is well known that a class of graphs $\mathcal{P}$ is hereditary if and only if $\mathcal{P} = Free(Y)$ for some set Y.

A property is said to be non-trivial if it contains at least one, but not all graphs. The *complementary property* of $\mathcal{P}$ is $\overline{\mathcal{P}} := \{\overline{G} \mid G \in \mathcal{P}\}$. Note that $\mathcal{P}$ is hereditary if and only if $\overline{\mathcal{P}}$ is. So a *co-additive hereditary* property, i.e. the complement of an additive hereditary property, is itself hereditary.

Let $\mathcal{P}_1, \ldots, \mathcal{P}_k$ be graph properties (classes) with $k > 1$. A graph $G = (V, E)$ is $(\mathcal{P}_1, \ldots, \mathcal{P}_k)$-*colorable* if there is a partition $(V_1, \ldots, V_k)$ of $V(G)$ such that $G[V_j] \in \mathcal{P}_j$ for each $j = 1, \ldots, k$. The problem of recognizing $(\mathcal{P}_1, \ldots, \mathcal{P}_k)$-colorable graphs is usually referred to as Generalized Graph Coloring [5]. When $\mathcal{P}_1 = \cdots = \mathcal{P}_k$ is the class $\mathcal{O}$ of edgeless graphs, this problem coincides with the standard k-COLORABILITY, which is known to be NP-complete for $k \geq 3$. Generalized Graph Coloring remains difficult for many other cases. For example, Cai and Corneil [7] showed that $(Free(K_n), Free(K_m))$-coloring is NP-complete for any integers $m, n \geq 2$, with the exception $m = n = 2$. This result, and others [1, 5, 14], have been recently generalized in [8] as follows.

Theorem 1 *If $\mathcal{P}_1, \ldots, \mathcal{P}_k$ ($k > 1$) are additive hereditary classes of graphs, then the problem of recognizing $(\mathcal{P}_1, \ldots, \mathcal{P}_k)$-colorable graphs is NP-hard, unless $k = 2$ and $\mathcal{P}_1 = \mathcal{P}_2$ is the class of edgeless graphs.*

Clearly, a similar result follows for co-additive properties. In the present paper we focus on the case where we have a mixture of additive and co-additive properties.

The *product* of graph classes $\mathcal{P}_1, \ldots, \mathcal{P}_k$ is $\mathcal{P}_1 \circ \cdots \circ \mathcal{P}_k := \{G \mid G \text{ is } (\mathcal{P}_1, \ldots, \mathcal{P}_k)\text{-colorable}\}$. A property is *reducible* if it is the product of two other properties, otherwise it is *irreducible*. It can be easily checked that the product of additive hereditary (or monotone) properties is again additive hereditary (respectively, monotone); and that $\overline{\mathcal{P}_1 \circ \cdots \circ \mathcal{P}_k} = \overline{\mathcal{P}_1} \circ \cdots \circ \overline{\mathcal{P}_k}$. So, without loss of generality we shall restrict our study to the case $k = 2$ and shall denote throughout the paper an additive property by $\mathcal{P}$ and co-additive by $\mathcal{Q}$. We will refer to the problem of recognizing $(\mathcal{P}, \mathcal{Q})$-colorable graphs as $(\mathcal{P} \circ \mathcal{Q})$-RECOGNITION.

The plan of the paper is as follows. In Section 2, we show that $(\mathcal{P} \circ \mathcal{Q})$-RECOGNITION cannot be simpler than $\mathcal{P}$- or $\mathcal{Q}$-RECOGNITION. In particular, we prove that $(\mathcal{P} \circ \mathcal{Q})$-RECOGNITION is NP-hard whenever $\mathcal{P}$- or $\mathcal{Q}$-RECOGNITION is NP-hard. Then, in Section 3, we study the problem under the assumption that both $\mathcal{P}$- and $\mathcal{Q}$-RECOGNITION are polynomial-time solvable and present infinitely many classes of $(\mathcal{P}, \mathcal{Q})$-colorable graphs with polynomial recognition time. These two results together give a complete answer to the question of complexity of $(\mathcal{P} \circ \mathcal{Q})$-RECOGNITION when $\mathcal{P}$ and $\overline{\mathcal{Q}}$ are additive monotone. When $\mathcal{P}$ and $\overline{\mathcal{Q}}$ are additive hereditary (but not both monotone), there remains an unexplored gap that we discuss in the concluding section of the paper.

2 NP-hardness

In this section we prove that if $\mathcal{P}$-RECOGNITION (or $\mathcal{Q}$-RECOGNITION) is NP-hard, then so is $(\mathcal{P} \circ \mathcal{Q})$-RECOGNITION. This is a direct consequence of the theorem below. In this theorem we use uniquely colorable graphs, which are often a crucial tool in proving coloring results.

A graph G is *uniquely* $(\mathcal{P}_1, \ldots, \mathcal{P}_k)$*-colorable* if $(V_1, \ldots, V_k)$ is its only $(\mathcal{P}_1, \ldots, \mathcal{P}_k)$-partition, up to some permutation of the V_i's. If, say, $\mathcal{P}_1 = \mathcal{P}_2$, then $(V_2, V_1, V_3, \ldots, V_k)$ will also be a $(\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \ldots, \mathcal{P}_k)$-coloring of G; such a permutation (of V_i's that correspond to equal properties) is a *trivial interchange*. A graph is *strongly uniquely* $(\mathcal{P}_1, \ldots, \mathcal{P}_k)$*-colorable* if $(V_1, \ldots, V_k)$ is the only $(\mathcal{P}_1, \ldots, \mathcal{P}_k)$-coloring, up to trivial interchanges.

When $\mathcal{P}_1, \ldots, \mathcal{P}_k$ are irreducible hereditary properties, and each $\mathcal{P}_i$ is either additive or co-additive, there is a strongly uniquely $(\mathcal{P}_1, \ldots, \mathcal{P}_k)$-colorable graph with each V_i non-empty. This important construction, for additive $\mathcal{P}_i$'s, is due to Mihók [16], with some embellishments by Broere and Bucko [4], while the proof of unique colorability follows from [10, Thm. 5.3]. Obviously, similar results apply to co-additive properties. The generalization to mixtures of additive and co-additive properties can be found in [9, Cor. 4.3.6, Thm. 5.3.2]

Theorem 2 *Let $\mathcal{P}$ and $\overline{\mathcal{Q}}$ be additive hereditary properties. Then there is a polynomial-time reduction from $\mathcal{P}$-RECOGNITION to $(\mathcal{P} \circ \mathcal{Q})$-RECOGNITION.*

Proof. Let $\mathcal{P} = \mathcal{P}_1 \circ \cdots \circ \mathcal{P}_n$ and $\mathcal{Q} = \mathcal{Q}_1 \circ \cdots \circ \mathcal{Q}_r$, where the $\mathcal{P}_i$'s and $\overline{\mathcal{Q}}_j$'s are irreducible additive hereditary properties. As noted above, there is a strongly uniquely $(\mathcal{P}_1, \ldots, \mathcal{P}_n, \mathcal{Q}_1, \ldots, \mathcal{Q}_r)$-colorable graph H with partition $(U_1, \ldots, U_n, W_1, \ldots, W_r)$, where each U_i and W_j is non-empty. Define $U := U_1 \cup \cdots \cup U_n$ and $W := W_1 \cup \cdots \cup W_r$. Arbitrarily fix a vertex $u \in U_1$, and define $N_W(u) := N(u) \cap W$. For any graph G, let the graph G_H consist of disjoint copies of G and H, together with edges $\{vw \mid v \in V(G), w \in N_W(u)\}$. We claim that $G_H \in \mathcal{P} \circ \mathcal{Q}$ if and only if $G \in \mathcal{P}$.

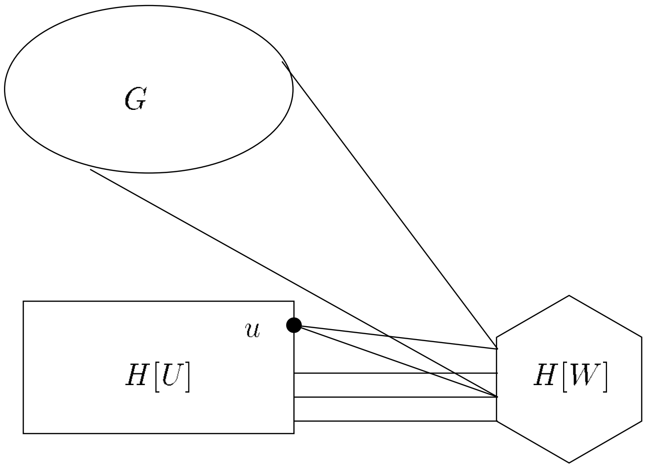

Figure 1: Using H to construct G_H.

If $G \in \mathcal{P}$, then, by additivity, $G \cup H[U]$ is in $\mathcal{P}$, and thus G_H is in $\mathcal{P} \circ \mathcal{Q}$. Conversely, suppose $G_H \in \mathcal{P} \circ \mathcal{Q}$, i.e. it has a $(\mathcal{P}_1, \ldots, \mathcal{P}_n, \mathcal{Q}_1, \ldots, \mathcal{Q}_r)$-partition, say $(X_1, \ldots, X_n, Y_1, \ldots, Y_r)$. Since H is strongly uniquely partitionable, we can assume that, for $1 \leq i \leq r$, $Y_i \cap V(H) = W_i$. Now, suppose for contradiction that, for some k, there is a vertex $v \in V(G)$ such that $v \in Y_k$; without loss of generality, let $k = r$. Then $G_H[W_r \cup \{v\}] \cong H[W_r \cup \{u\}]$ is in $\mathcal{Q}_r$, so $(U_1 \setminus \{u\}, U_2, \ldots, U_n, W_1, \ldots, W_{r-1}, W_r \cup \{u\})$ is a new $(\mathcal{P}_1, \ldots, \mathcal{P}_n, \mathcal{Q}_1, \ldots, \mathcal{Q}_r)$-partition of H, which is impossible. Thus, $V(G) \subseteq X_1 \cup \cdots \cup X_n$, and hence $G \in \mathcal{P}$, as claimed.

Since H is a fixed graph, G_H can be constructed in time linear in $|V(G)|$, so the theorem is proved. ■

3 Polynomial time results

Lemma 1 *For any $\mathcal{P} \subseteq Free(K_n)$ and $\mathcal{Q} \subseteq Free(\overline{K}_m)$, there exists a constant $\tau = \tau(\mathcal{P}, \mathcal{Q})$ such that for every graph $G = (V, E) \in \mathcal{P} \circ \mathcal{Q}$ and every subset $B \subseteq V$ with $G[B] \in \mathcal{P}$, at least one of the following statements holds:*

(a) there is a subset $A \subseteq V$ such that $G[A] \in \mathcal{P}$, $G[V - A] \in \mathcal{Q}$, and $|A - B| \leq \tau$,

(b) there is a subset $C \subseteq V$ such that $G[C] \in \mathcal{P}$, $|C| = |B| + 1$, and $|B - C| \leq \tau$.

Proof. By the Ramsey Theorem [13], for each positive integers m and n, there is a constant $\tau(m,n)$ such that every graph with more than $\tau(m,n)$ vertices contains either a $\overline{K}_m$ or a K_n as an induced subgraph. For two classes $\mathcal{P} \subseteq Free(K_n)$ and $\mathcal{Q} \subseteq Free(\overline{K}_m)$, we define $\tau = \tau(\mathcal{P},\mathcal{Q})$ to be equal $\tau(m,n)$. Let us show that with this definition the proposition follows.

Let $G=(V,E)$ be a graph in $\mathcal{P} \circ \mathcal{Q}$, and B a subset of V such that $G[B] \in \mathcal{P}$. Consider an arbitrary subset $A \subseteq V$ such that $G[A] \in \mathcal{P}$ and $G[V-A] \in \mathcal{Q}$. If (a) does not hold, then $|A-B| > \tau$. Furthermore, $G[B-A] \in \mathcal{P} \cap \mathcal{Q} \subseteq Free(K_n, \overline{K}_m)$, and hence $|B-A| \le \tau$. Therefore, $|A| > |B|$. But then any subset $C \subseteq A$ such that $A \cap B \subseteq C$ and $|C| = |B|+1$ satisfies (b). ■

Lemma 1 suggests the following recognition algorithm for graphs in the class $\mathcal{P} \circ \mathcal{Q}$.

Algorithm $\mathcal{A}$

Input: A graph $G=(V,E)$.
Output: **YES** if $G \in \mathcal{P} \circ \mathcal{Q}$, or **NO** otherwise.

(1) Find in G any inclusion-wise maximal subset $B \subseteq V$ inducing a K_n-free graph.

(2) If there is a subset $C \subseteq V$ satisfying condition (b) of Lemma 1, then set $B := C$ and repeat Step (2).

(3) If G contains a subset $A \subseteq V$ such that

$$|B-A| \le \tau,$$

$$|A-B| \le \tau,$$

$$G[A] \in \mathcal{P},$$

$$G[V-A] \in \mathcal{Q},$$

output **YES**, otherwise output **NO**.

Theorem 3 *If graphs on p vertices in a class $\mathcal{P} \subseteq Free(K_n)$ can be recognized in time $O(p^k)$ and graphs in a class $\mathcal{Q} \subseteq Free(\overline{K}_m)$ can be recognized in time $O(p^l)$, then Algorithm $\mathcal{A}$ recognizes graphs on p vertices in the class $\mathcal{P} \circ \mathcal{Q}$ in time $O(p^{2\tau+\max\{(k+2),\max\{k,l\}\}})$, where $\tau = \tau(\mathcal{P},\mathcal{Q})$.*

Proof. Correctness of the algorithm follows from Lemma 1. Now let us estimate its time complexity. In Step (2), the algorithm examines at most $\binom{p}{\tau}\binom{p}{\tau+1}$ subsets C and for each of them verifies whether $G[C] \in \mathcal{P}$ in time $O(p^k)$. Since Step (2) loops at most p times, its time complexity is $O(p^{2\tau+k+2})$. In Step (3), the algorithm examines at most $\binom{p}{\tau}^2$ subsets A, and for each A, it verifies whether $G[A] \in \mathcal{P}$ in time $O(p^k)$ and whether $G[V-A] \in \mathcal{Q}$ in time $O(p^l)$. Summarizing, we conclude that the total time complexity of the algorithm is $O(p^{2r+\max\{(k+2),\max\{k,l\}\}})$. ■

Notice that Theorem 3 generalizes several positive results on the topic under consideration. For instance, the split graphs [12], which are $(Free(K_2), Free(\overline{K_2}))$-colorable by definition, can be recognized in polynomial time. More general classes have been studied under the name of polar graphs in [6, 15, 17]. By definition, a graph is $(m-1, n-1)$ polar if it is $(\mathcal{P}, \mathcal{Q})$-colorable with $\mathcal{P} = Free(K_n, P_3)$ and $\mathcal{Q} = Free(\overline{K}_m, \overline{P}_3)$. It is shown in [15] that for any particular values of $m \geq 2$ and $n \geq 2$, $(m-1, n-1)$ polar graphs on p vertices can be recognized in time $O(p^{2m+2n+3})$.

Further examples generalizing the split graphs were examined in [2] and [11], where the authors showed that classes of graphs partitionable into at most two independent sets and two cliques can be recognized in polynomial time. These are special cases of $(\mathcal{P} \circ \mathcal{Q})$-RECOGNITION with $\mathcal{P} \subseteq Free(K_3)$ and $\mathcal{Q} \subseteq Free(\overline{K}_3)$.

4 Concluding results and open problems

Theorems 2 and 3 together provide complete answer to the question of complexity of $(\mathcal{P} \circ \mathcal{Q})$-RECOGNITION in case of monotone properties $\mathcal{P}$ and $\overline{\mathcal{Q}}$. Indeed, if $\mathcal{P}$ is an additive monotone non-trivial property, then $\mathcal{P} \subseteq Free(K_n)$ for a certain value of n, since otherwise it includes all graphs. Similarly, if $\overline{\mathcal{Q}}$ is additive monotone, then $\mathcal{Q} \subseteq Free(\overline{K}_m)$ for some m. Hence, the following theorem holds.

Theorem 4 *If $\mathcal{P}$ and $\overline{\mathcal{Q}}$ are additive monotone properties, then $(\mathcal{P} \circ \mathcal{Q})$-RECOGNITION has polynomial-time complexity if and only if $\mathcal{P}$- and $\mathcal{Q}$-RECOGNITION are both polynomial-time solvable; moreover, $(\mathcal{P} \circ \mathcal{Q})$-RECOGNITION is in NP if and only if $\mathcal{P}$- and $\mathcal{Q}$-RECOGNITION are both in NP.*

If $\mathcal{P}$ and $\overline{\mathcal{Q}}$ are general additive hereditary properties (not necessarily monotone), then there is an unexplored gap containing properties $\mathcal{P} \circ \mathcal{Q}$, where $\mathcal{P}$ and $\mathcal{Q}$ can both be recognized in polynomial time, but $\mathcal{K} \subset \mathcal{P}$ or $\mathcal{O} \subset \mathcal{Q}$ (where $\mathcal{K} := \overline{\mathcal{O}}$ is the set of cliques). In the rest of this section we show that this gap contains both NP-hard and polynomial-time solvable instances, and propose several open problems to study.

For a polynomial time result we refer the reader to [17], where the authors claim that $(\mathcal{P} \circ \mathcal{Q})$-RECOGNITION is polynomial-time solvable if $\mathcal{P}$ is the class of edgeless graphs and $\mathcal{Q} = Free(\overline{P}_3)$. Notice that $Free(\overline{P}_3)$ contains all edgeless graphs and hence Theorem 3 does not apply to this case. Interestingly enough, when we extend $\mathcal{P}$ to the class of bipartite graphs, we obtain an NP-hard instance of the problem, as the following theorem shows.

Theorem 5 *If $\mathcal{P}$ is the class of bipartite graphs and $\mathcal{Q} = Free(\overline{P}_3)$, then $(\mathcal{P} \circ \mathcal{Q})$*-RECOGNITION *is NP-hard.*

Proof. We reduce the standard 3-COLORABILITY to our problem. Consider an arbitrary graph G and let G' be the graph obtained from G by adding a triangle $T = (1, 2, 3)$ with no edges between G and T. We claim that G is 3-colorable if and only if G' is $(\mathcal{P}, \mathcal{Q})$-colorable.

First, assume that G is 3-colorable and let V_1, V_2, V_3 be a partition of $V(G)$ into three independent sets. We define $V_j' = V_j \cup \{j\}$ for $j = 1, 2, 3$. Then $G'[V_1' \cup V_2']$ is a bipartite graph and $G'[V_3'] \in Free(\overline{P}_3)$, and the proposition follows.

Conversely, let $U \cup W$ be a partition of $V(G')$ with $G'[U]$ being a bipartite graph and $G'[W] \in Free(\overline{P}_3)$. Clearly, $T \nsubseteq U$. If $T - U$ contains a single vertex, then $G'[W - T]$ is an edgeless graph, since otherwise a $\overline{P}_3$ arises. If $T - U$ contains more than one vertex, then $W - T = \emptyset$ for the same reason. Clearly, in both cases G is a 3-colorable graph. ■

This discussion presents the natural question of exploring the boundary that separates polynomial from non-polynomial time solvable instances in the above-mentioned gap. As one of the smallest classes in this gap with unknown recognition time complexity, let us point out $(\mathcal{P}, \mathcal{Q})$-COLORABLE graphs with $\mathcal{P} = \mathcal{O}$ and $\mathcal{Q} = Free(2K_2, P_4)$, where $2K_2$ is the disjoint union of two copies of K_2.

Another direction for prospective research deals with $(\mathcal{P}, \mathcal{Q})$-colorable graphs where $\mathcal{P}$ or $\mathcal{Q}$ is neither additive nor co-additive. This area seems to be almost unexplored and also contains both NP-hard and polynomial-time solvable problems. To provide some examples, let $\mathcal{Q}$ be the class of complete bipartite graphs, which is obviously neither additive nor co-additive. The class of graphs partitionable into an independent set and a complete bipartite graph has been studied in [3] under the name of bisplit graphs and has been shown there to be polynomial-time recognizable. Again, extension of $\mathcal{P}$ to the class of all bipartite graphs transforms the problem into an NP-hard instance.

Theorem 6 *If $\mathcal{P}$ is the class of bipartite graphs and $\mathcal{Q}$ is the class of complete bipartite graphs, then $(\mathcal{P} \circ \mathcal{Q})$*-RECOGNITION *is NP-hard.*

Proof. The reduction is again from 3-COLORABILITY. For a graph G, we define G' to be the graph obtained from G by adding a new vertex adjacent to every vertex of G. It is a trivial exercise to verify that G is 3-colorable if and only if G' is $(\mathcal{P}, \mathcal{Q})$-COLORABLE. ■

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