Orientable convexity, geodetic and hull numbers in graphs

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Abstract

We prove three results conjectured or stated by Chartrand, Fink and Zhang [European J. Combin **21** (2000) 181–189, Disc. Appl. Math. **116** (2002) 115–126, International Journal of Mathematics and Mathematical Sciences **36** (2003) 2265–2275.]: a connected graph has orientations with different geodetic numbers, orientations with different hull numbers, and, if there are no end-vertices, orientations with different convexity numbers. The proof of the first result is a correction of Chartrand and Zhang's proof, and allows for an easy proof of the second result. The third result says roughly that graphs without end-vertices can be oriented anti-transitively.

Key words: oriented graph, convex, geodesic, hull number, transitively orientable

The aim of this paper is to establish the following results, for every connected graph G with at least three vertices:

$$g^-(G) < g^+(G) \tag{1}$$
$$h^-(G) < h^+(G) \tag{2}$$
$$con^-(G) < con^+(G) \text{ iff } G \text{ has no end-vertices.} \tag{3}$$

Results (2) and (3) were conjectured by Chartrand, Fink and Zhang in [3] and [2], respectively. The first result was stated by Chartrand and Zhang in [1, Thm. 2.5], but there was a gap in their proof. They independently noticed this gap, and an alternative proof was found, but the correction we present in Section 3 allows us to prove (1) and (2) simultaneously. We prove (3) in Section 2.

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1 Preliminaries

Let $D = (V, A)$ be a digraph, and let u and v be vertices. A $u-v$ *geodesic* is a dipath from u to v with the least possible number of arcs (so all $u-v$ geodesics have the same length). The *closed interval* $I[u, v]$ consists of u, v, and every vertex that is on some $u - v$ geodesic or on some $v - u$ geodesic (note that there may be no directed path at all from u to v, or from v to u). For a set $S \subseteq V(D)$, we define $I[S] := \bigcup_{u,v \in S} I[u, v]$, and, for $k > 0$, $I^k[S] := I[I^{k-1}[S]]$, where $I^0[S] := S$.

A set S is *convex* if $S = I[S]$, that is, every geodesic between every two vertices of S lies in S. The *convex hull* $[S]$ of S is the smallest convex set containing S; this is the intersection of all convex sets containing S, and also the limit of the sequence $S \subseteq I[S] \subseteq I^2[S] \subseteq \cdots$.

A *hull-set* of D is a set $S \subseteq V$ for which $[S] = V$. If, moreover, $I[S] = V$, then S is a *geodetic set*. The *hull number* of D is

$$h(D) := \min\{|S| \mid S \text{ is a hull-set of } D\},$$

while the *geodetic number* of D is

$$g(D) := \min\{|S| \mid S \text{ is a geodetic set of } D\}.$$

For an undirected graph G, an orientation $\overrightarrow{G}$ is a digraph obtained by giving each edge one of its two possible directions. The *lower* and *upper orientable hull numbers* are, respectively,

$$\begin{aligned} h^-(G) &:= \min\{h(\overrightarrow{G}) \mid \overrightarrow{G} \text{ is an orientation of } G\}, \text{ and} \\ h^+(G) &:= \max\{h(\overrightarrow{G}) \mid \overrightarrow{G} \text{ is an orientation of } G\}. \end{aligned}$$

The *lower* and *upper orientable geodetic numbers* $g^-(G)$ and $g^+(G)$ are defined similarly.

Let v be a vertex in a digraph $D = (V, A)$. Its *in-* and *out-neighbourhood* are $N^-(v) := \{u \mid \overrightarrow{uv} \in A\}$ and $N^+(v) := \{w \mid \overrightarrow{vw} \in A\}$, respectively. Its *in-* and *out-degree* are $id(v) := |N^-(v)|$ and $od(v) := |N^+(v)|$, respectively. If, for every $u \in N^-(v)$ and every $w \in N^+(v)$, $\overrightarrow{uw} \in A$, then v is *extreme*. An (extreme) vertex v is a *source* if $N^-(v) = \emptyset$, and a *sink* if $N^+(v) = \emptyset$.

A graph that can be oriented so that every vertex is extreme is a *comparability* graph, or *transitively orientable* graph. A result that we will use repeatedly is the following, due to Chartrand et al. [2, Prop. 2.1], [3, Prop. 1.3]:

1. Proposition. *A vertex v is extreme iff, for every u and w in V, v is not an interior vertex of any $u - w$ geodesic. Therefore, v is extreme iff $V - v$ is a convex set, iff v is contained in every hull-set and every geodetic-set.* $\square$

2 Orientable convexity numbers

If $D = (V, A)$ is a digraph, the *convexity number* $con(D)$ is the size of the largest convex set $C \subsetneq V$ (V itself is always convex). For an undirected graph G, $con^-(G)$ and $con^+(G)$ are the minimum and maximum convexity numbers over all orientations of G. We are interested in whether $con^-(G) < con^+(G)$.

By Proposition 1, D has an extreme vertex iff $con(D) = n - 1$, where n is the number of vertices. For any graph G, we can make an arbitrary vertex v extreme by orienting all incident edges away from v, so we always have $con^+(G) = n - 1$. Moreover, if G contains an end-vertex x, then in every orientation x is either a source or a sink; so in this case, $con^-(G) = n - 1$ too.

If G has no end-vertices, it is straightforward to find an orientation with no sources or sinks; the reader is encouraged to do so, and then try to generalise this to avoid all extreme vertices. We present a solution below.

Let some of the edges of G be oriented. A vertex incident to some oriented edge is an *or-vertex*, short for *oriented vertex*. Note that a vertex v is non-extreme iff there are arcs $\overrightarrow{uv}$ and $\overrightarrow{vw}$, such that uw is either not present, or it is already oriented as $\overleftarrow{uw}$. No matter how the remaining undirected edges are oriented, v remains non-extreme.

2. Theorem. *A graph with minimum degree 2 can be oriented so that all its vertices are non-extreme. Thus, for a connected graph G with at least 3 vertices, $con^-(G) < con^+(G)$ iff G has no end-vertices.*

Proof: Since G has minimum degree 2, it contains a cycle. Find a maximal set M of edge-disjoint *chordless* cycles, and orient their edges to make them directed cycles. We claim that every or-vertex v is now non-extreme. If v is on a triangle uvw in M, then $\overrightarrow{uv}$, $\overrightarrow{vw}$ and $\overrightarrow{wu}$ are all arcs. Otherwise, v is on a chordless cycle in M of length at least 4, with neighbours, say, u and w, where $uw \notin E(G)$ but $\overrightarrow{uv}$ and $\overrightarrow{vw}$ are arcs.

We now show that, if there are unoriented vertices, we can orient one or more while maintaining the property that all or-vertices are non-extreme.

Any unoriented vertex u must be on a path $u_0, \dots, u_{r+1}$ joining distinct or-vertices u_0 and u_{r+1} (because the graph has minimum degree at least 2, and our initial set M of edge-disjoint cycles was chosen to be maximal). Taking r to be as small as possible ensures that the internal vertices $u_1, \dots, u_r$ are all unoriented. Directing the path as $\overrightarrow{u_0u_1}$, ..., $\overrightarrow{u_ru_{r+1}}$ ensures that $u_1, \dots, u_r$ all have positive in- and out-degree. Moreover, if $r > 1$, then, for $1 \leq i \leq r$, $u_{i-1}u_{i+1} \notin E(G)$, and thus u_i is non-extreme.

If $r = 1$, then we might have to orient differently if u_0u_2 is an edge of G. Since u_0u_1 and u_1u_2 are unoriented, and M was maximal, u_0u_2 must be oriented, say as $\overrightarrow{u_0u_2}$. Now orienting u_0u_1 and u_1u_2 as $\overleftarrow{u_0u_1}$ and $\overleftarrow{u_1u_2}$, ensures that u_1 is on a directed triangle and is thus non-extreme. $\square$

3 Orientable geodetic and hull numbers

Chartrand and Zhang's proof of (1) essentially found a vertex v_1, and orientations D_1 and D_2 of G, such that if S is a hull-set in D_2, then $I_{D_2}[S] \subseteq I_{D_1}[S - v_1]$ (this is Claim 1 in our own proof). Moreover, v_1 was a source in D_2, and was thus contained in every hull-set. By taking S to be a minimum geodetic set for D_2, we immediately get $g^-(G) < g^+(G)$. With slightly more work (Claim 2 in our proof), we also get $h^-(G) < h^+(G)$, proving Conjecture 3.10 of [3].

Chartrand and Zhang stated their result only for orientable geodetic numbers, as they did not include Claim 2. Moreover, they oriented $G[U]$ arbitrarily (where U is a set of vertices specified in the proof, and $G[U]$ is the subgraph of G induced by U). The path of length four (for example) shows that this does not always work, and their alternative proof did not extend to showing $h^-(G) < h^+(G)$. There is, however, an orientation of $G[U]$ that will rescue the original proof, as we show below.

3. Theorem. *For any connected graph G with at least three vertices, $g^-(G) < g^+(G)$ and $h^-(G) < h^+(G)$.*

Proof: If G is a complete graph with vertices $v_1, \dots, v_n$, we first orient G transitively (that is, $v_i \to v_j$ iff $i < j$). Since every vertex is extreme, this orientation shows that $g^+(G) = n = h^+(G)$. Reversing the orientation of v_1v_2, ..., $v_{n-1}v_n$ makes $\{v_1, v_2\}$ a geodetic set; thus $g^-(G) = 2 = h^-(G)$.

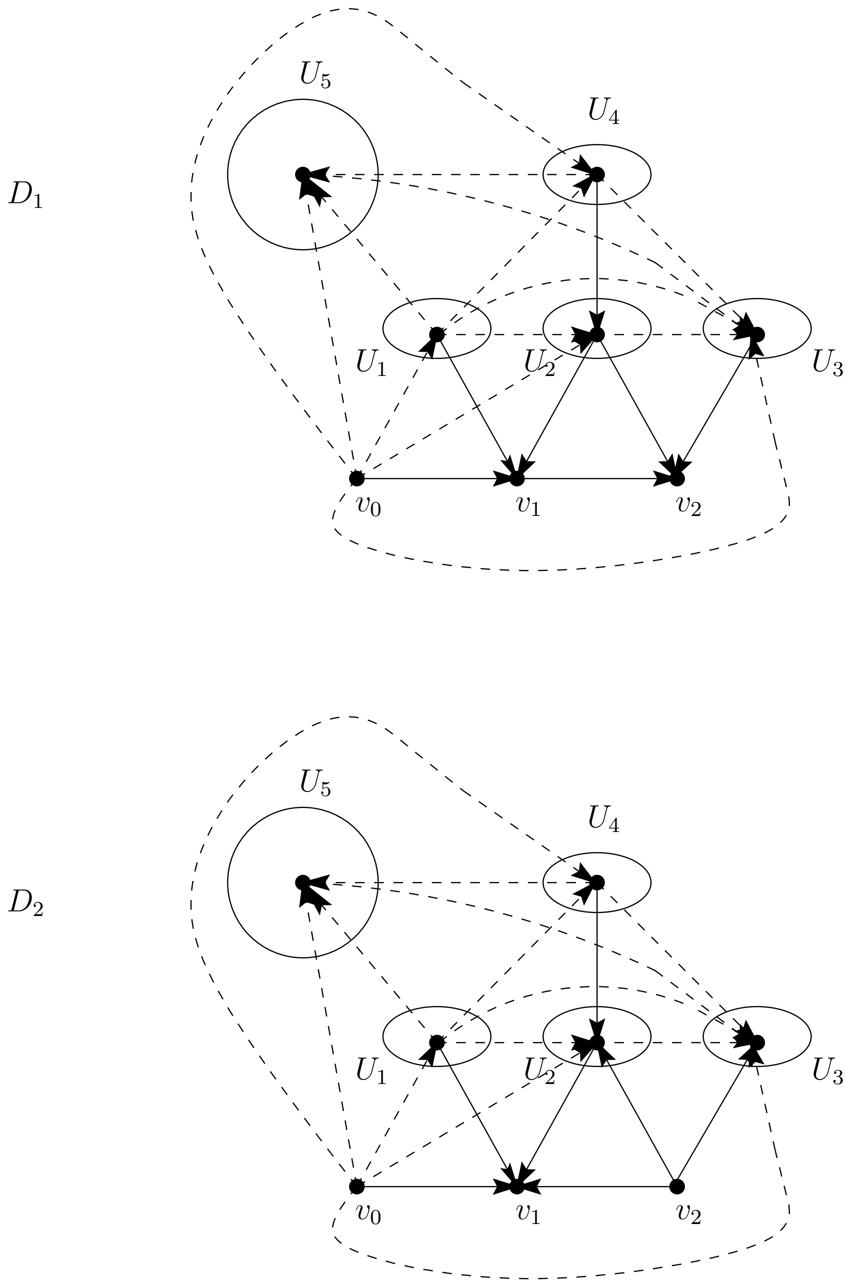

Fig. 1. The orientations D_1 and D_2 of G.

If G is not complete, then we can find vertices v_0, v_1, v_2 that induce a path of length two. Figure 1 shows all the adjacencies (solid lines) and possible adjacencies (dashed lines) in G, where the U_i's are defined as follows. For a set $C \subseteq V(G)$, $N(C)$ is the set $\{v \in V \mid \exists c \in C, vc \in E\}$.

$$
\begin{aligned}
U &:= V(G) \setminus \{v_0, v_1, v_2\},\\
U_1 &:= U \cap (N(v_1) \setminus N(v_2)),\\
U_2 &:= U \cap (N(v_1) \cap N(v_2)),\\
U_3 &:= U \cap (N(v_2) \setminus N(v_1)),\\
U_4 &:= (U \cap N(U_2)) \setminus (U_1 \cup U_2 \cup U_3), \text{ and}\\
U_5 &:= U \setminus (U_1 \cup U_2 \cup U_3 \cup U_4).
\end{aligned}
$$

Let D_2 be the digraph obtained by orienting G as follows[1]. We orient an edge xy from x to y if one of the following conditions holds:

$$
\begin{array}{ll}
x \in \{v_0, v_2\} & \text{or} \quad y = v_1,\\
x \in U_1 & \text{and } y \in U \setminus U_1,\\
x \in U_4 & \text{and } y \in U_2,\\
x \in U \setminus U_3 & \text{and } y \in U_3.
\end{array}
$$

All other edges join vertices within the same U_i, and are oriented arbitrarily. It can be checked that the conditions are self-consistent. We obtain D_1 from D_2 by reversing the orientation of the arcs incident to v_2.

Claim 1: If S is a hull-set in D_2, then $I_{D_2}[S] \subseteq I_{D_1}[S - v_1]$.

Since S is a hull-set for D_2, it must contain the extreme vertices v_0 and v_2. In D_1, v_1 is on a $v_0 - v_2$ geodesic, and is thus in $I_{D_1}[S - v_1]$. So $S \subseteq I_{D_1}[S - v_1]$.

Consider, therefore, a vertex $w \in I_{D_2}[S] \setminus S$; note that $w \in U$. This vertex must be an internal vertex of an $a - b$ geodesic P in D_2, for some a and b in S. If a and b are both in U, then $V(P) \subseteq U$; since the orientation of $G[U]$ is the same in D_1 as in D_2, P is present in D_1. Moreover, the $a - b$ dipaths in D_1 are just the $a - b$ dipaths in D_2, so P is still a *shortest* $a - b$ dipath. Since a and b are in $S - v_1$, $w \in I_{D_1}[S - v_1]$.

If $a = v_0$, then $b \neq v_1$ (since the only $v_0 - v_1$ geodesic is $\overrightarrow{v_0v_1}$), and clearly $b \neq v_2$, so $b \in U$. Moreover, the $a - b$ dipaths do not use v_1 or v_2, so D_1 contains all the $a - b$ dipaths of D_2, and no others; thus P is still an $a - b$ geodesic in D_1. As above, a and b are in $S - v_1$, so $w \in I_{D_1}[S - v_1]$.

If $a = v_2$, then b must be in $N(v_2)$; but then the unique $a - b$ geodesic in D_2 is $\overrightarrow{ab}$, with no internal vertices.

[1] The labeling is chosen to be consistent with Chartrand and Zhang, but I prefer to describe D_2 before D_1.

If $b = v_1$, then I claim that P must have vertices awv_1, with $a \in U_4$ and $w \in U_2$. To see this, note that a cannot be in $N(v_1)$, as otherwise the only $a - v_1$ geodesic is $\overrightarrow{av_1}$. Moreover, there are no dipaths from $U_3 \cup U_5$ to v_1, so a must be in U_4. By definition of U_4, and by the choice of orientation, there is a (directed) path of length two from a to v_1, so every $a - v_1$ geodesic has length two. The internal vertex must be adjacent to v_1, but cannot be in U_1 (by choice of orientation), so it must be in U_2.

Since a is in U_4, it is not adjacent to v_2; but in D_1 there is a directed path awv_2, and this is therefore an $a - v_2$ geodesic. Since a and v_2 are in $S - v_1$, w is in $I_{D_1}[S - v_1]$.

Claim 2: If S is a hull-set in D_2, then $I^{\ell}_{D_2}[S] \subseteq I^{\ell}_{D_1}[S - v_1]$ for any $\ell \geq 1$.

We proceed by induction on ℓ, the base case $\ell = 1$ following from Claim 1. Now for $\ell > 1$,

$$\begin{aligned} I^{\ell}_{D_2}[S] = I_{D_2}[I^{\ell-1}_{D_2}[S]] &\subseteq I_{D_1}[I^{\ell-1}_{D_2}[S] - v_1] \subseteq \\ &\subseteq I_{D_1}[I^{\ell-1}_{D_1}[S - v_1] - v_1] \subseteq I_{D_1}[I^{\ell-1}_{D_1}[S - v_1]] = I^{\ell}_{D_1}[S - v_1]. \end{aligned}$$

The first containment follows from Claim 1 applied to the hull-set $I^{\ell-1}_{D_2}[S]$, while the second follows from the inductive hypothesis.

If S is a hull-set for D_2, then $I^{k}_{D_2}[S] = V$, for some k. By Claim 2, $I^{k}_{D_1}[S - v_1] = V$, so $S - v_1$ is a hull-set for D_1. In particular, v_1 is a sink in D_2, so it is contained in S, and taking S to be a minimum hull-set for D_2 we have

$$h^{-}(G) \leq h(D_1) \leq |S - v_1| < |S| = h(D_2) \leq h^{+}(G).$$

If S is a (minimum) geodetic set for D_2, then we can take $k = 1$, so $S - v_1$ is a geodetic set for D_1 and we have $g^{-}(G) < g^{+}(G)$. $\square$

Since every geodetic set is a hull-set, we have $h(D) \leq g(D)$ for every digraph D. For an undirected graph G we therefore have $h^{-}(G) \leq g^{-}(G)$ and $h^{+}(G) \leq g^{+}(G)$, and together with Theorem 3 this leaves five possibilities:

$$h^{-} = g^{-} < h^{+} = g^{+} \tag{4}$$
$$h^{-} = g^{-} < h^{+} < g^{+} \tag{5}$$
$$h^{-} < g^{-} < h^{+} = g^{+} \tag{6}$$
$$h^{-} < g^{-} = h^{+} < g^{+} \tag{7}$$
$$h^{-} < h^{+} < g^{-} < g^{+}. \tag{8}$$

Chartrand et al. identified many infinite classes of graphs for which (4) holds, including trees, cycles and complete bipartite graphs. For complete bipartite graphs $K_{s,t}$ with $s \geq t \geq 2$ [1, Prop. 3.8], and for transitively orientable graphs with a Hamiltonian path, we have $h^-(G) = g^-(G) = 2 < n = h^+(G) = g^+(G)$. If T is a tree with k end-vertices, then $h^-(T) = g^-(T) = k < |V(T)| = h^+(T) = g^+(T)$, while $h^-(C_{2n+1}) = g^-(C_{2n+1}) = 2 < 2n = h^+(C_{2n+1}) = g^+(C_{2n+1})$. We leave the realisability of (5) – (8) as open problems.

4. Problem. *Find infinite classes of graphs for which (5), (6) or (7) hold. Are there (infinitely many) graphs for which (8) holds?*

Note that (8) cannot hold for graphs G for which there is an orientation $\overrightarrow{G}$ such that $g(\overrightarrow{G}) = h(\overrightarrow{G})$. However, there are probably many graphs for which no such orientation exists.

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