Dually Vertex-Oblique Graphs

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Abstract

A vertex with neighbours of degrees $d_1 \geq \cdots \geq d_r$ has *vertex type* $(d_1, \ldots, d_r)$. A graph is *vertex-oblique* if each vertex has a distinct vertex-type (no graph can have distinct degrees). Schreyer, Walther and Mel'nikov [Vertex-oblique graphs, same proceedings] have constructed infinite classes of *super vertex-oblique* graphs, where the degree-types of G are distinct even from the degree types of $\overline{G}$.

G is vertex-oblique iff $\overline{G}$ is; but G and $\overline{G}$ cannot be isomorphic, since self-complementary graphs always have non-trivial automorphisms. However, we show by construction that there are *dually vertex-oblique graphs* of order n, where the vertex-type sequence of G is the same as that of $\overline{G}$; they exist iff $n \equiv 0$ or $1 \pmod 4), n \geq 8$, and for $n \geq 12$ we can require them to be split graphs.

We also show that a dually vertex-oblique graph and its complement are never the unique pair of graphs that have a particular vertex-type sequence; but there are infinitely many super vertex-oblique graphs whose vertex-type sequence is unique.

1 Introduction and basic results

Let G be a simple graph on n vertices. A vertex v of degree r, with neighbours of degrees $x_1 \geq \cdots \geq x_r$, has *vertex type* $t(v) := (x_1, \ldots, x_r)$. G is *vertex-oblique* if each vertex has a distinct vertex-type.

It is known [12] that G is vertex-oblique if and only if $\overline{G}$ is, as we now show. Let the degree-set of G (the set of degrees of vertices of G, including repetitions) be $\mathcal{D}$. The degree-sets of v, and its neighbourhood, are $\{r\}$ and $\mathcal{N}_v := \{x_1, \ldots, x_r\}$, respectively, where $r = |\mathcal{N}_v|$. The non-neighbourhood of v therefore has degree-set $\mathcal{N'}_v := (\mathcal{D} \setminus \mathcal{N}_v) \setminus \{|\mathcal{N}_v|\}$; let this degree-set be $y_1 \leq \cdots \leq y_{n-1-r}$ when listed in non-decreasing order.

The degree of v in $\overline{G}$ (the complement of G) is therefore $\overline{r} := |\mathcal{N}'_v| = n-1-r$, and its vertex-type in $\overline{G}$ is $\overline{t}(v) = (\overline{y_1}, \ldots, \overline{y_{\overline{r}}})$, which is completely determined by $t(v)$ and $\mathcal{D}$. Since $\mathcal{D}$ is the same for all vertices, the $t(v)$'s are distinct iff the $\overline{t}(v)$'s are distinct.

While no graph can have distinct degrees, Schreyer et al. [12] have constructed infinite classes of vertex-oblique graphs. In fact, their examples are *super vertex-oblique*, with the degree-types of G being distinct even from degree types of $\overline{G}$.

It is natural to ask whether there are any self-complementary vertex-oblique graphs, but this is impossible because a self-complementary graph always has non-trivial automorphisms [9, 11], obtained by applying twice an(y) isomorphism that maps the graph to its complement. However, in this article we construct infinitely many *dually vertex-oblique graphs*, where the set of vertex-types of G is the same as that of $\overline{G}$.

Many simple results for self-complementary graphs still hold under the weaker assumption that G and $\overline{G}$ have the same set of vertex-types. In particular, dually vertex-oblique graphs of order n can only exist if $n \equiv 0$ or $1 \pmod 4$. The main result of this paper, proved in Sections 2, 3, is that they exist for all feasible n at least 8:

1. Theorem. *Dually vertex-oblique graphs of order $n > 1$ exist iff n is congruent to 0 or $1 \pmod 4$, and $n \geq 8$.* □

We will make use of the following elementary lemma that is inspired by similar results on self-complementary graphs.

2. Lemma. *Let G be a graph with the same degree sequence as $\overline{G}$, say $d_1 \geq \cdots \geq d_n$. Then:*

A. $d_i + d_{n-i+1} = n - 1$, *for* $i = 1, \ldots, n$.

B. $n \equiv 0$ *or* $1 \pmod 4$.

If, moreover, G has the same set of vertex-types as $\overline{G}$, and there are r_d vertices of degree d, and $s_{x,y}$ edges joining vertices of degrees x and y, then:

C. *$s_{y,y}+s_{\overline{y},\overline{y}} = \binom{r_y}{2}$, and if $x \neq y$, $s_{x,y}+s_{\overline{x},\overline{y}} = r_x r_y$; in particular, $s_{d,\overline{d}} = \frac{1}{2}r_d^2$ for all d, except if $d = \overline{d} = (n-1)/2$ in which case $s_{d,\overline{d}} = \frac{1}{2}\binom{r_d}{2}$.*

D. *r_d is even for all d, except (if $n \equiv 1 \pmod 4$) for $r_{(n-1)/2} \equiv 1 \pmod 4$.*

Furthermore, if G is dually vertex-oblique, then:

E. *$r_d < 2d, d \neq (n-1)/2$; in particular, there are no isolated or end-vertices.*

F. *There must be at least three different degrees in G.*

Proof: A. If r_d vertices have degree d in G, then r_d vertices have degree d in $\overline{G}$ and, thus, degree $n-1-d$ in G. So the degree sequence is symmetric about $\frac{1}{2}(n-1)$.
B. The number of edges of G and $\overline{G}$ is the same: $\frac{1}{2}\binom{n}{2} = \frac{1}{4}n(n-1)$, which must be an integer.
C. Since $s_{x,y}$ is determined by the vertex-types of G, it must remain the same in $\overline{G}$, that is, $s_{x,y} = \overline{s}_{x,y}$; similarly, $s_{\overline{x},\overline{y}} = \overline{s}_{\overline{x},\overline{y}}$. Now a vertex of degree x (or y) in G has degree $\overline{x}$ (or $\overline{y}$) in $\overline{G}$. So if there are p unordered pairs of vertices $\{v,w\}$ with $d(v) = x$ and $d(w) = y$, then $p - s_{x,y} = \overline{s}_{\overline{x},\overline{y}} = s_{\overline{x},\overline{y}}$. Since $p = \binom{r_y}{2}$ when $x = y$, and $p = r_x r_y$ otherwise, the result follows.
D. By C, $\frac{1}{2}r_d^2$ must be an integer, so r_d is even for $d \neq \overline{d}$. When there are vertices of degree $(n-1)/2$, by B we must have $n = 4k+1$ for some k. By the first part, there are $2r$ vertices with degree $d < (n-1)/2$, and therefore $2r$ vertices with degree $d > (n-1)/2$, leaving $(4k+1) - 2(2r) = 4(k-r)+1$ vertices of degree $(n-1)/2$.
E. If G is vertex-oblique, the vertices of degree d cannot all be adjacent to vertices of degree $\overline{d}$ only. So $\frac{1}{2}r_d^2 = s_{d,\overline{d}} < dr_d$.
F. Clearly G cannot be regular, so there must be at least two different degrees. If n is odd, the number of different degrees must be odd, by A. Suppose n is even and there are exactly two degrees, say $d < (n-1)/2$ and $\overline{d} > (n-1)/2$. Since G is vertex-oblique, each vertex v of degree d must be adjacent to a distinct number n_v of vertices of degree $\overline{d}$. Note that $0 \leq n_v \leq d$. If $n_v = 0$ for some v, then in the complement v would be a vertex of degree $\overline{d}$ that is adjacent to all vertices of degree d, so $\overline{G}$ would have no vertex w of degree d with $n_w = 0$. Similar reasoning excludes the case $n_v = d$, so we have

$0 < n_v < d$ for every v. This means that there are less than $d < n/2 = r_d$ possible values of n_v, a contradiction. $\square$

By Lemma 2.B, the smallest possible orders $n > 1$ for a dually vertex-oblique graph are 4 and 5; but by part E these could not have any vertices of degree $d < (n-1)/2$, so no such graphs exist when $n < 8$. We now construct such graphs for every $n \equiv 0$ or $1 \pmod 4$, $n \geq 8$.

2 Construction on $4k$ vertices

We show in this section that dually vertex-oblique graphs exist for all $n \equiv 0 \pmod 4, \mathrm{n} \geq 8$. This proves half of Theorem 1, the other half being proved in the (much shorter) subsequent section.

A dually vertex-oblique graph on 8 vertices is shown in Figure 1, with the degree and vertex-type displayed next to each vertex. To verify this, one has to check that for every vertex with vertex-type $(x_1, \ldots, x_r)$, there is another vertex that is non-adjacent to vertices of degrees $(7 - x_1, \ldots, 7 - x_r)$.

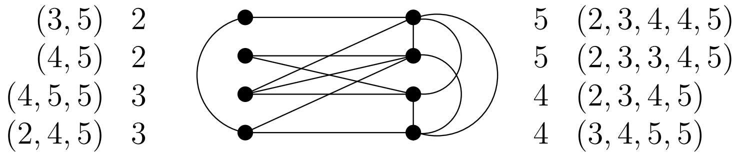
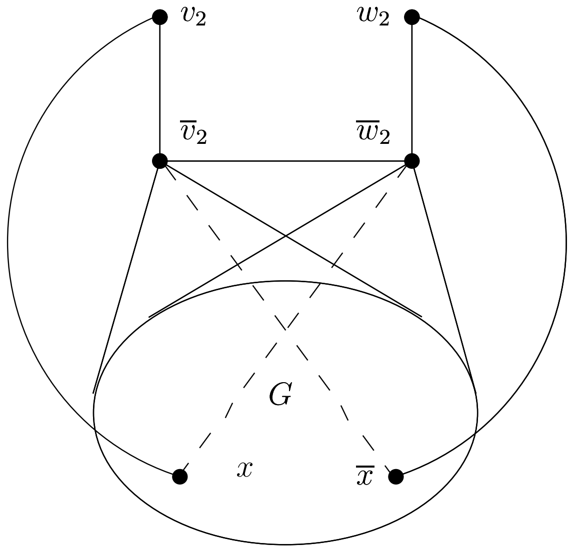


Figure 1: A dually vertex-oblique graph on 8 vertices.

Given a dually vertex-oblique graph G on $n = 4k$ vertices, we now show how to construct G' on $n+4 = 4k'$ vertices, where $k' := k+1$ (see Figure 2). We add vertices v_2, w_2, that will have degree 2, and $\overline{v}_2, \overline{w}_2$ that will have degree $4k' - 3 = 4k + 1$. Moreover, the new vertices induce a P_4, in a manner reminiscent of Akiyama and Harary's [1] method of producing larger self-complementary graphs.

We pick an arbitrary vertex $x \in V(G)$, and let $\overline{x}$ be the (unique) vertex such that $\overline{t}(\overline{x}) = t(x)$. Note that if x has degree d_x, then $\overline{x}$ has degree $4k - 1 - d_x \neq d_x$. We make v_2 adjacent to $\overline{v}_2$ and x, w_2 adjacent to $\overline{w}_2$ and $\overline{x}$. Meanwhile, $\overline{v}_2$ is adjacent to $v_2, \overline{w}_2$ and $V(G) \setminus \overline{x}$; and $\overline{w}_2$ is adjacent to $w_2, \overline{v}_2$ and $V(G) \setminus x$.



(dashed lines indicate non-adjacency)

Figure 2: Larger dually vertex-oblique graphs from smaller ones.

A vertex of degree d now has degree $d' := d + 2$; this also means that a vertex of the complementary degree $\overline{d} := 4k - 1 - d$ now still has complementary degree $\overline{d'} = 4k' - 1 - d' = \overline{d} + 2$. The degrees in $V(G)$ now range between at least 4 and at most $4k - 1$ (by Lemma 2.E), and thus the degrees (and vertex-types) in $V(G)$ are distinct from those of the new vertices.

If $u \notin \{x, \overline{x}\}$ had vertex-type $t(u) = (d_1, \ldots, d_r)$, in G' it has type $t'(u) = (d'_1, \ldots, d'_r, 4k' - 3, 4k' - 3)$. The unique vertex $\overline{u}$ such that $\overline{t}(\overline{u}) = t(u)$ was non-adjacent in G to vertices of degrees $\overline{d_1}, \ldots, \overline{d_r}$; so in G' it is non-adjacent to vertices of degrees $\overline{d'_1}, \ldots, \overline{d'_r}$, as well as two vertices of degree $2 = \overline{4k' - 3}$.

Meanwhile $t(x) = \overline{t}(\overline{x}) = (f_1, \ldots, f_d)$ becomes $t'(x) = \overline{t'}(\overline{x}) = (2, f_1 + 2, \ldots, f_d + 2, 4k' - 3)$. Distinct vertex-types in G therefore result in distinct vertex-types in G', and complementary vertex-types result in complementary vertex-types. Moreover, $t(v_2) = \overline{t}(\overline{v}_2) = (d_{G'}(x), 4k' - 3)$, and $t(w_2) = \overline{t}(\overline{w}_2) = (d_{G'}(\overline{x}), 4k' - 3)$, so the types of the new vertices are also distinct and complementary, and thus G is dually vertex-oblique.

A graph G is *split* if its vertex set partitions into $\mathcal{L} \cup \mathcal{R}$ (the "left" and "right" vertices), where $G[\mathcal{L}]$ is edgeless and $G[\mathcal{R}]$ is complete. Our constructions are close to being split graphs, with the vertex-set partitioning into vertices of degree less than $2k$ and vertices of degree at least $2k$. With a little more effort we can construct examples (for $n \geq 12$) that are actually split graphs.

We will construct an appropriate bipartite graph with partition $\mathcal{L} \cup \mathcal{R}$ and show that, if we add edges to make $\mathcal{R}$ induce a clique, the resulting graph is dually vertex-oblique. If B is bipartite, with partition $\mathcal{L} \cup \mathcal{R}$, its *bipartite complement* is the graph $\widetilde{B}$ with $V(\widetilde{B}) := V(B)$, $E(\widetilde{B}) := \{uv \mid u \in \mathcal{L}, v \in \mathcal{R}, uv \notin E(B)\}$. The vertex-type of v in $\widetilde{B}$ is $\widetilde{t}(v)$. A *dually semi-vertex-oblique* graph is a bipartite graph B with $\mathcal{L} = \{\ell_1, \ldots, \ell_{2k}\}, \mathcal{R} = \{r_1, \ldots, r_{2k}\}$, such that:

(i) $\{t(\ell_1), \ldots, t(\ell_{2k})\}$ contains no repetitions

(ii) $\{t(\ell_1), \ldots, t(\ell_{2k})\} = \{t(r_1), \ldots, t(r_{2k})\}$

(iii) $\{\widetilde{t}(\ell_1), \ldots, \widetilde{t}(\ell_{2k})\} = \{t(\ell_1), \ldots, t(\ell_{2k})\}$, and (thus) $\{\widetilde{t}(r_1), \ldots, \widetilde{t}(r_{2k})\} = \{t(r_1), \ldots, t(r_{2k})\}$.

If ℓ_1, say, had degree $2k$, then in $\widetilde{B}$ it would have degree 0, so by conditions (ii) and (iii) there must be a vertex of degree 0 in $\mathcal{R}$, a contradiction. So the

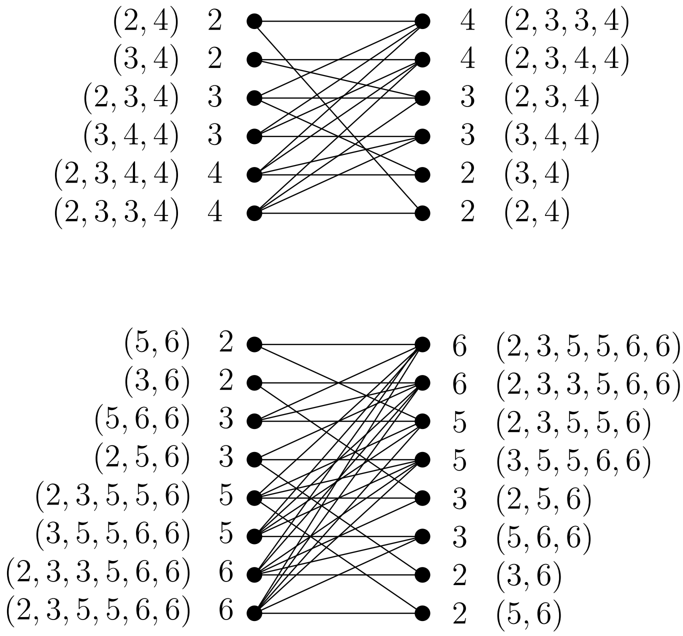


Figure 3: Dually semi-vertex-oblique graphs on 12 and 16 vertices.

minimum degree in $\mathcal{L}$ is at least 1, and by (iii) the maximum degree is at most $2k-1$, and similarly for $\mathcal{R}$.

We now add edges to make $\mathcal{R}$ induce a clique, giving us a split graph G. The degree of any vertex r_j jumps up by $2k-1$, so its degree becomes at least $2k$; thus the degrees (and vertex-types) of vertices in $\mathcal{R}$ become distinct from those in $\mathcal{L}$. If $t(r_i)$ differed from $t(r_j)$ in the number of entries equal to d, where $d < 2k$, then in G both types get $2k-1$ new entries that are all at least $2k$, but they still differ in the number of entries equal to d. If $t(\ell_i)$ differed from $t(\ell_j)$ in the number of entries equal to d, in G they will differ in the number of entries equal to $d+2k-1$. Thus G is vertex-oblique, and from (iii) we can see that $\overline{G}$ has the same vertex-types as G.

Dually semi-vertex-oblique graphs on 12 and 16 vertices are shown in Figure 3, with their degrees and vertex-types. To verify condition (iii), one has to check that for every vertex on the left with vertex-type $(x_1, \ldots, x_r)$, there is another vertex that is non-adjacent to vertices (on the right) of degrees $(2k-x_1, \ldots, 2k-x_r)$.

Given a dually semi-vertex-oblique graph B on $n = 4k$ vertices, we now

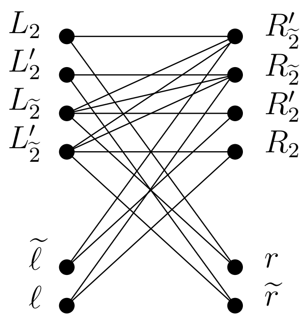


Figure 4: Adding vertices to make larger semi-vertex-oblique graphs.

show how to construct B' on $n+8 = 4k'$ vertices, where $k' := k+2$. By (iii) and (i), respectively, we can find two vertices $r, \widetilde{r} \in \mathcal{R}$ such that $t(r) = \widetilde{t}(\widetilde{r})$, and $deg(r) \neq k$ (so that $deg(r) \neq deg(\widetilde{r})$). By (ii) there are (unique) vertices $\ell, \widetilde{\ell} \in \mathcal{L}$ with $t(\ell) = t(r), t(\widetilde{\ell}) = t(\widetilde{r})$.

We add vertices $L_2, L_{\widetilde{2}}, L'_2, L'_{\widetilde{2}}$ to $\mathcal{L}$, and $R_2, R_{\widetilde{2}}, R'_2, R'_{\widetilde{2}}$ to $\mathcal{R}$. Their adjacencies are defined below, and sketched in Figure 4. The vertices with subscript $\widetilde{2}$ are adjacent to all vertices not shown on the other side, giving them degree $2k'-2 = 2k+2$; while the vertices with subscript 2 have degree 2.

$$
\begin{array}{lcr}
N(L_2) & := & \{R'_{\widetilde{2}}, r\} \\
N(L'_2) & := & \{R_{\widetilde{2}}, \widetilde{r}\} \\
N(L_{\widetilde{2}}) & := & \mathcal{R} \setminus \{R_2, \widetilde{r}\} \\
N(L'_{\widetilde{2}}) & := & \mathcal{R} \setminus \{R'_2, r\}
\end{array}
\qquad
\begin{array}{lcr}
N(R_2) & := & \{L'_{\widetilde{2}}, l\} \\
N(R'_2) & := & \{L_{\widetilde{2}}, \widetilde{l}\} \\
N(R_{\widetilde{2}}) & := & \mathcal{L} \setminus \{L_2, \widetilde{l}\} \\
N(R'_{\widetilde{2}}) & := & \mathcal{L} \setminus \{L'_2, l\}
\end{array}
$$

These adjacencies are well-defined, and the construction is symmetric with respect to $\mathcal{L}$ and $\mathcal{R}$ (as far as vertex-types are concerned), so (ii) holds.

The degrees of every ℓ_i increase by 2 (so they now range between at least 3 and at most $2k+1$). If $t(\ell_i)$ differed from $t(\ell_j)$ in the number of entries equal to d, in B' they will differ in the number of entries equal to $d+2$. Besides, $t(L_2), t(L'_2), t(L_{\widetilde{2}})$ and $t(L'_{\widetilde{2}})$ are distinct from each other (because $deg(r) \neq deg(\widetilde{r})$), and from the $t(\ell_i)$'s (because $2 < deg(\ell_i) < 2k+2$, for all i). Thus (i) holds.

Condition (iii) is still valid for the ℓ_i's, while the new vertices' neighbourhoods are as follows in the bipartite complement $\widetilde{B}$:

$$
\begin{array}{rclrcl}
N_{\widetilde{B}}(L_2) & = & \mathcal{R} \setminus \{R'_{\widetilde{2}}, r\} & N_{\widetilde{B}}(R_2) & = & \mathcal{L} \setminus \{L'_{\widetilde{2}}, l\} \\
N_{\widetilde{B}}(L'_2) & = & \mathcal{R} \setminus \{R_{\widetilde{2}}, \widetilde{r}\} & N_{\widetilde{B}}(R'_2) & = & \mathcal{L} \setminus \{L_{\widetilde{2}}, \widetilde{l}\} \\
N_{\widetilde{B}}(L_{\widetilde{2}}) & = & \{R_2, \widetilde{r}\} & N_{\widetilde{B}}(R_{\widetilde{2}}) & = & \{L_2, \widetilde{l}\} \\
N_{\widetilde{B}}(L'_{\widetilde{2}}) & = & \{R'_2, r\} & N_{\widetilde{B}}(R'_{\widetilde{2}}) & = & \{L'_2, l\}
\end{array}
$$

Since $deg_{\widetilde{B}}(\widetilde{r}) = deg_B(r)$ and $deg_{\widetilde{B}}(R_{\widetilde{2}}) = deg_{\widetilde{B}}(R'_{\widetilde{2}}) = deg_B(R_2) = deg_B(R'_2)$, we have $\widetilde{t}(L_{\widetilde{2}}) = t(L_2)$ and $\widetilde{t}(L'_{\widetilde{2}}) = t(L'_2)$.

3 Construction on $4k + 1$ vertices

We now conclude the proof of Theorem 1, showing that dually vertex-oblique graphs exist for all $n \equiv 1 \pmod 4, \mathrm{n} \geq 8$. Take a dually vertex-oblique graph G on $4k$ vertices, and introduce a new vertex u_0 that is adjacent to the $2k$ vertices with degree $d \geq 2k$. We claim that the resulting graph G' of order $n' = 4k + 1$ is again dually vertex-oblique. Note that if G was a split graph, then the high-degree vertices must have formed the clique, and thus G' will also be split.

A vertex of degree $d := d_G(v)$ in G has degree $d' := d_{G'}(v)$ in G'. So $d' = d$ if $d < 2k$, and $d' = d + 1$ if $d \geq 2k$. If v and w had complementary degrees in G, that is, $d_G(v) + d_G(w) = n - 1 = 4k - 1$, then $d_{G'}(v) + d_{G'}(w) = n' - 1 = 4k$, so v and w still have complementary degrees in G'; this means that if $\overline{d} = f$ in G, then $\overline{d'} = f'$ in G'. Also, $d_{G'}(u_0) + d_{G'}(u_0) = 2k + 2k = n' - 1$.

In G', u_0 will be the unique vertex of degree $2k$; and if v, w, were adjacent in G to different numbers of vertices of degree d, then in G' they are adjacent to different numbers of vertices of degree d'; so G' is vertex-oblique.

Since G is dually vertex-oblique, for every vertex v there is a unique vertex $\overline{v}$ with $\overline{t}(\overline{v}) = t(v)$. If $t(v) = (x_1, \ldots, x_r)$, then $\overline{v}$ must have non-neighbours in G of degrees $\overline{x_1}, \ldots, \overline{x_r}$. If $r < 2k$, then u_0 is adjacent to $\overline{v}$ but not to v, so in G' v has vertex-type $t'(v) = (x'_1, \ldots, x'_r)$, and $\overline{v}$ has non-neighbours of degrees $\overline{x'_1}, \ldots, \overline{x'_r}$; thus $\overline{t}'(\overline{v}) = t'(v)$. If $r > 2k$ then u_0 is adjacent to v in G', and to $\overline{v}$ in $\overline{G'}$; thus $t'(v) = (x'_1, \ldots, 2k, \ldots, x'_r) = \overline{t}'(\overline{v})$. Finally $t'(u_0) = \overline{t}'(u_0)$, so G' is dually vertex-oblique.

4 Vertex-type sequences: uniqueness and non-uniqueness

Some graphs, such as complete graphs, edgeless graphs and matchings, have unique degree sequence (that is, no other graph has the same degree sequence) and, thus, unique vertex-type sequence. If G is dually vertex-oblique, then by definition its complement shares the same vertex-type sequence, and is not isomorphic to G because self-complementary graphs have non-trivial automorphisms. But could this complementary pair be the unique graphs with that vertex-type sequence? We show here that the answer is always 'No', but that there are infinitely many super vertex-oblique graphs with unique vertex-type sequence. We give some definitions first.

The *degree sequence* of a graph on n vertices is the sequence $d_1 \geq \cdots \geq d_n$ of its degrees. The *vertex-type sequence* is the sequence $t_1 \succeq \cdots \succeq t_n$ of vertex-types, where $t_i \succ t_j$ if t_i is longer than t_j, or if t_i and t_j have the same length and t_i is lexicographically larger than t_j. G_d is the subgraph of G induced by vertices of degree d, and (for $p \neq q$) $G_{p,q}$ is the bipartite subgraph induced by edges joining a vertex of degree p to a vertex of degree q.

A *switch* is the replacement of edges v_0w_0, v_1w_1, with new edges v_0w_1, v_1w_0 (that is, v_0w_1 and v_1w_0 did not appear in the original graph); this does not change the degree of any vertex, but may change the vertex-types. A (d, d')-*switch* (or just 'restricted switch', when d and d' are not specified) is a switch where v_0 and v_1 both have degree d, and w_0 and w_1 both have degree d' (possibly equal to d); such a switch does not change the type of any vertex.

3. Theorem. *For any dually vertex-oblique graph G, there is a graph $H \notin \{G, \overline{G}\}$ with the same vertex-type sequence as G.*

Proof: We will establish:

Claim. For any degree $d \neq (n-1)/2$, G has distinct vertices v_0, v_1, of degree d, and w_0, w_1, of degree $\overline{d}$, such that $v_0w_0, v_1w_1 \in E(G)$, $v_0w_1, v_1w_0 \notin E(G)$.

The result follows from the claim since we can then perform a $(d, \overline{d})$-

switch which gives us another graph H without changing the type of any vertex. Since G has trivial automorphism group, $H \not\cong G$, and we will show that $H \not\cong \overline{G}$.

Let x be any vertex not in $\{v_0, v_1, w_0, w_1\}$. Let $\overline{x}$ be the unique vertex that has the same vertex-type in $\overline{G}$ as x has in G. If $\overline{x} \neq x$ (possibly $\overline{x} \in \{v_0, v_1, w_0, w_1\}$), note that x and $\overline{x}$ are adjacent in H iff they are adjacent in G iff they are *not* adjacent in $\overline{G}$. If $x = \overline{x}$, take another vertex $y \notin \{x, v_0, v_1, w_0, w_1\}$; note that x and y exist by the remark after Lemma 2. If $y \neq \overline{y}$ we are done, otherwise note that x and y are adjacent in H iff they are adjacent in G iff $\overline{x} = x$ and $\overline{y} = y$ are not adjacent in $\overline{G}$.

We now turn to proving the Claim, which is equivalent to saying that the bipartite graph $G_{d,\overline{d}}$ has an induced $2K_2$. If there is any vertex z of degree d that is adjacent to no (or all) vertices of degree $\overline{d}$, then in $\overline{G}$ z would be a vertex of degree $\overline{d}$ adjacent to all (or no) vertices of degree d, contradicting the fact that G and $\overline{G}$ have the same vertex-type sequence. So in $G_{d,\overline{d}}$ every vertex has at least one neighbour and one non-neighbour from the opposite part.

Let $G_{d,\overline{d}}$ have bipartition $D \cup \overline{D}$. In what follows, x_i will be a vertex in D, $N_i \subseteq \overline{D}$ the set of its neighbours, and $N'_i := \overline{D} \setminus N_i$; $N_i \neq \emptyset \neq N'_i$ by the previous argument. Take an arbitrary vertex $x_0 \in D$. Pick a vertex $y_1 \in N'_0$, and let y_1 be adjacent to some vertex x_1; clearly $x_1 \neq x_0$. If there is a vertex $\widetilde{y}_0 \in N_0$ such that $\{x_0, x_1, \widetilde{y}_0, y_1\}$ induce a $2K_2$, we are done; otherwise, x_1 is adjacent to all of N_0, as well as y_1, so $N_1 \supset N_0$, and $N'_1 \subset N'_0$. Pick a vertex $y_2 \in N'_1$, and let y_2 be adjacent to x_2; as before, $x_2 \neq x_1$, and either there is $\widetilde{y}_1 \in N_1$ such that $\{x_1, x_2, \widetilde{y}_1, y_2\}$ induce a $2K_2$, or $N'_2 \subset N'_1$. Repeating this procedure we must eventually find an induced $2K_2$, since N'_i can never be empty. $\square$

By contrast, it can be shown that there are infinitely many super vertex-oblique graphs with unique vertex-type sequence. We will use the following result, whose second part was discovered by Zofia Majcher and generalised by Igor É. Zverovich.

4. Theorem [3, 5] [7, 8, 13]. *If G, H, are two labeled graphs with the same degree sequence, then G can be obtained from H by a sequence of switches. If G and H even have the same vertex-type sequence, then G can*

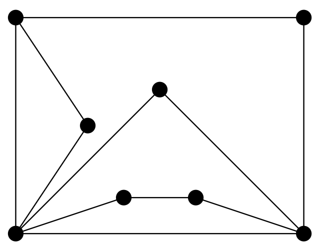


Figure 5: Schreyer & Walther's super vertex-oblique graph X with unique vertex-type sequence.

be obtained from H by restricted switches only. □

5. Corollary. *If no G_d and no $G_{p,q}$ contains an induced $2K_2$, then G has unique vertex-type sequence.* □

The converse of the corollary is not true (the matchings are a counterexample), because a restricted switch may give us a graph G' isomorphic to G. But it follows from Corollary 5 that if every degree appears at most once in some graph G, except for some degree that appears at most three times, then G has unique vertex-type sequence. It can also be used to show that some of the super-vertex-oblique graphs discovered by Jens Schreyer and Hansjoachim Walther (private communication) have unique vertex-type sequence.

In particular, every degree appears exactly once in their graph X (our notation - see Fig. 5) except for five vertices of the same degree that induce a graph with only one edge. These properties are preserved if we add a vertex y adjacent to the unique vertex v of maximum degree, and a vertex z adjacent to $y \cup (X \setminus v)$. This is Construction 1 (with $k = 1$) from their paper [12], where they prove that the resulting graph is again super vertex-oblique. Since we can apply this procedure repeatedly, always preserving these properties, we have:

6. Corollary. *There are infinitely many connected super vertex-oblique graphs with unique vertex-type sequence.* □

5 Recognising degree and vertex-type sequences

A graph G *realises* its degree sequence, and its vertex-type sequence. Several authors have given different characterisations of degree sequences of graphs[1]. Erdős and Gallai [4] showed that a sequence $d_1 \geq \cdots \geq d_n$ is realised by some graph if and only if, for $r = 1, \ldots, n-1$, we have

$$\sum_{i=1}^{r} d_i \leq r(r-1) + \sum_{j=r+1}^{n} \min(r, d_j).$$

If G is a graph with the same degree sequence as $\overline{G}$, and r_d is the number of vertices of degree d, then:

(*) $d_i + d_{n-i+1} = n-1$, for $i = 1, \ldots, n$;

(**) r_d is even for all d, except for $r_{(n-1)/2} \equiv 1 \pmod 4$.

Clapham and Kleitman [2] showed by construction that every sequence that satisfies (*), (**) and the Erdős-Gallai conditions, is realised by a self-complementary graph. However, not all such sequences are realised by a dually vertex-oblique graph: for every $n \equiv 0$ or $1 \pmod 4$, there are self-complementary graphs with end-vertices [1] but no such dually vertex-oblique graphs (Lemma 2.E). It would be interesting to characterise the degree sequences of dually vertex-oblique graphs. One might also ask similar questions about vertex-type sequences:

7. Problem. *A. When is a sequence (of sequences of positive integers) the vertex-type sequence of some graph?*
B. Characterise the degree sequences and vertex-type sequences of:

- *vertex-oblique graphs,*
- *super vertex-oblique graphs, and*
- *dually vertex-oblique graphs.*

[1]We also realised after submitting the paper that many of the observations in this section were made by Zofia Majcher in [7, 8].

Z. Majcher [7, 8] noted that the Erdős-Gallai results on degree-sequences, together with the Gale-Ryser conditions explained below, lead to an efficient algorithm to solve the vertex-type sequence problems; if the sequence is realised by some graph, the algorithm can also be made to construct an example. However, we would like a more succinct characterisation similar to that of Erdős-Gallai, Clapham-Kleitman or Gale-Ryser, especially as this might shed light on the degree sequence problems.

Gale [6] and Ryser [10] showed that sequences $p_1 \geq \cdots \geq p_m$ and $q_1 \geq \cdots \geq q_n$ are the degrees of a bipartite graph B (with the p_i's being degrees on one side, and the q_j's the degrees on the other side) if and only if, for $r = 1, \ldots, n-1$:

$$\sum_{i=1}^{m} \min\{r, p_i\} \geq \sum_{j=1}^{r} q_j.$$

Given the vertex-type sequence of a graph G, we can recover the degree sequence, and compute the vertex-types of $\overline{G}$ (as noted at the beginning of the introduction); it is then straightforward to check whether G is (super or dually) vertex-oblique. So we turn our attention to Problem A.

If we want to check whether a given sequence is actually the vertex-type sequence of some graph G, we recover the degree-sequences of the G_d's and $G_{p,q}$'s (for all d, p, q, in the degree-sequence of G), and check the Erdős-Gallai and Gale-Ryser conditions, respectively. If the conditions are not all satisfied, we have a contradiction; otherwise, we can construct G_d's and $G_{p,q}$'s that together give us a graph with the given vertex-type sequence.

6 Other open problems

In a self-complementary graph of order $4k + 1$, one can always remove an appropriate vertex to get a self-complementary graph of order $4k$. It is not clear whether an analogous claim is true for dually vertex-oblique graphs.

8. Problem. *Is there a dually vertex oblique graph on $4k + 1$ vertices, such that removing any vertex of degree $2k$ leaves a subgraph H such that (a) H does not have the same vertex-types as its complement, or (b) H is not vertex-oblique, or both (a) and (b)?*

For any fixed k, Schreyer et al. [12] constructed super vertex-oblique graphs that were k-connected, with k-connected complements. Our examples of dually vertex-oblique graphs have vertices of degree 2, and thus connectivity at most 2.

9. Problem. *Are there (complementary pairs of) dually vertex-oblique graphs of arbitrarily high connectivity?*

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