New results on self-complementary graphs : regular self-complementary graphs.

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Abstract

In 1977 Kotzig [Selected open problems in graph theory, in *Graph Theory and Related Topics* (1979) 358–367] posed a number of problems that spurred a lot of research into regular self-complementary graphs.

The last two authors found a family of counterexamples to one of his conjectures; here we expand this family further. We also use a graph found by Hartsfield [On regular self-complementary graphs, *J. Graph Theory* **11** (1987) 537–538] as a counterexample to another of Kotzig's conjectures, to construct infinite families of self-complementary graphs for which none of the isomorphisms between G and $\overline{G}$ have uniform cycle lengths.

In response to yet another of Kotzig's problems, Rao [On regular and strongly-regular self-complementary graphs, *Discrete Math.* **54** (1985) 73–82] characterised vertex-transitive self-complementary graphs in terms of their orbits. We extend one of his results to digraphs and then bring together this theorem, and the work of 27 other authors, to classify regular self-complementary graphs and digraphs into two grand hierarchies.

In 1977 Kotzig [16] posed a number of problems that spurred a lot of research into regular self-complementary graphs. We explain the problems in Section 1. A counterexample to one of the conjectures, found by Hartsfield, is extended to an infinite family in Section 2. In Section 3 a family of counterexamples to one of the conjectures (found by the last two authors) is expanded further.

In response to yet another of Kotzig's problems, Rao [31] found a lot of important information about the orbits of self-complementary graphs, and used this to characterise the vertex-transitive self-complementary graphs. We extend his results to digraphs in Theorem 7. We end by using Rao's result, our extension, and the work of 23 other authors, to classify regular self-complementary graphs and digraphs into two grand hierarchies (see Figure 4).

1 Basic results and definitions

Throughout this article, we use n and m to denote the number of vertices and edges, respectively, of a graph G. If G is isomorphic to its complement $\overline{G}$ we call it a *self-complementary graph*, or *sc-graph* for short. The isomorphisms between G and $\overline{G}$ are called *antimorphisms* and can be looked at as permutations of $V(G)$.

1. Lemma. *Let G be a self-complementary graph on n vertices. Then*

A. $|E(G)| = \frac{n(n-1)}{4}$, *and thus $n \equiv 0$ or $1 \pmod 4)$.*

B. *If v is a vertex of degree r in G, any antimorphism must map it to a vertex of degree $n-1-r$ in G; thus the degree sequence of G must be symmetric about $\frac{1}{2}(n-1)$.*

C. *In particular, a regular sc-graph must have $4k+1$ vertices and degree $2k$ for some k.* □

The following result was obtained independently by Sachs [36] and Ringel [32]:

2. Theorem. *An antimorphism σ consists of cycles whose lengths are divisible by 4, and a single fixed vertex whenever $n = 4k+1$. We call the fixed vertex a trivial cycle.* □

For any graph G, and any vertex v, we define $t(v)$ and $\overline{t}(v)$ to be the number of triangles of G or $\overline{G}$, respectively, which contain v. For self-complementary graphs we define the following sets:

$$\begin{aligned}
\hat{F}(G) &:= \{u \in V(G) | t(u) = \overline{t}(u)\}, \\
F(G) &:= \{u \in V(G) | \text{ for some antimorphism } \sigma, \sigma(u) = u\}, \text{ and} \\
N(G) &:= \{uv \in E(G) | \text{ for some antimorphism } \sigma, \sigma(u) = v\}.
\end{aligned}$$

Sachs' and Ringel's papers were the first ever published on self-complementary graphs. A number of others appeared in later years, among them one by Kotzig [16] that included six problems on regular sc-graphs; this acted directly as a catalyst for some of the further research done in this area. Here we shall be concerned with the following problems:

3. Kotzig's Problems.

A. *Is it true that every regular self-complementary graph has an antimorphism where all cycles are of length 4, except for a single fixed vertex?*

B. *Characterise the set $F(G)$ of a regular sc-graph G. Is it true that $F(G) = \hat{F}(G)$ for a regular sc-graph G?*

C. *Characterise the set $N(G)$ for a sc-graph G.*

D. *Is is true that a regular sc-graph G is strongly regular iff $F(G) = V(G)$ and $N(G) = E(G)$?*

2 Antimorphisms with unequal cycle lengths

Kotzig's first problem asked whether every regular sc-graph G has an antimorphism with all non-trivial cycles having length 4. In general we can ask whether every sc-graph (regular or not) has at least one antimorphism with non-trivial cycles of equal length. However, Hartsfield [13] answered both questions in the negative, giving a regular graph H (Figure 1) with 13 vertices and four antimorphisms, which all have a 1-cycle, a 4-cycle and an 8-cycle:

$$\begin{aligned}
\sigma &= (1)(2\,3\,4\,5)(6\,7\,8\,9\,10\,11\,12\,13) \\
\sigma^3 &= (1)(2\,5\,4\,3)(6\,9\,12\,7\,10\,13\,8\,11) \\
\sigma^5 &= (1)(2\,3\,4\,5)(6\,11\,8\,13\,10\,7\,12\,9) \\
\sigma^7 &= (1)(2\,5\,4\,3)(6\,13\,12\,11\,10\,9\,8\,7).
\end{aligned}$$

We note that the vertex 1 is the only fixed vertex of H, and its neighbourhood is $N = \{2, 4, 6, 8, 10, 12\}$. We define $\overline{N} := \{3, 5, 7, 9, 11, 13\}$.

Four years after Hartsfield, Xu [39] gave a regular graph on 13 vertices whose antimorphisms include $\sigma := (1)(7\,4\,10\,12\,13\,6\,8\,11\,5\,3\,2\,9)$. Xu mentioned σ^{11} too, but for *any* antimorphism and *any* odd i, σ^i is also an antimorphism. So the graph does have antimorphisms (σ^3 and σ^9) whose cycles are all of length 4. It is thus not a counterexample to Kotzig's conjecture.

However, Zhou [44] later gave a correct construction of regular sc-graphs of every order $4k + 1 \geq 17$, each of whose antimorphisms has 0 or 1 cycles of length 4, depending on whether k is even or odd.

Before we learnt of Zhou's result, we constructed another infinite class, based on that of Hartsfield.

First we observe that if we remove the vertex 1 we get a counterexample of order 12. To see this, note that any antimorphism of $H - \{1\}$ must exchange the vertices of N and

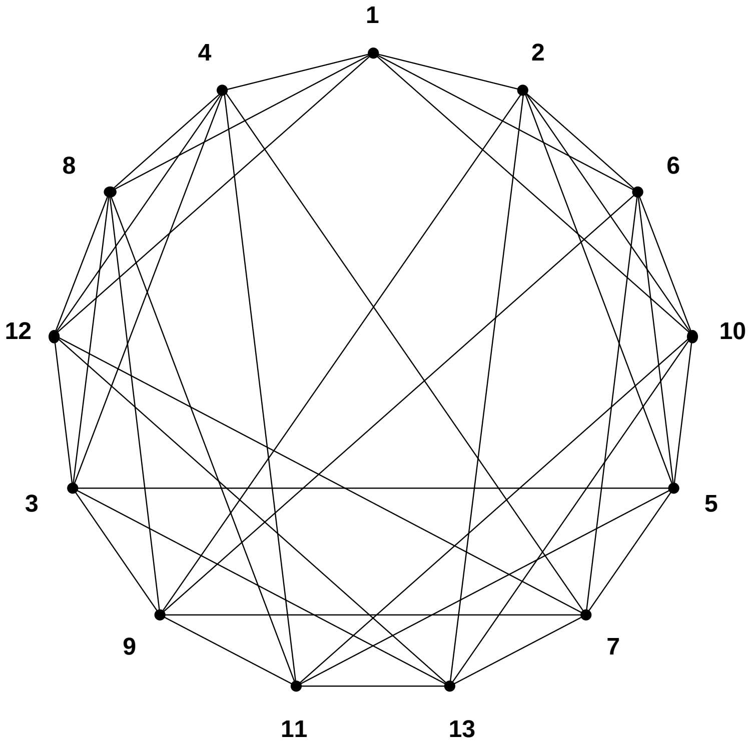

Figure 1: Hartsfield's graph.

$\overline{N}$ because they have distinct degrees (cf. Lemma 1.B). So the antimorphisms of $H - \{1\}$ can be extended to an antimorphism of H and, thus, must have cycles of length 4 and 8.

Now, if J is any counterexample of order n, we can add a path on 4 vertices, with the inner two vertices adjacent to all the vertices of J (see Figure 2). This standard construction [1] produces a new graph J' with 4 more vertices. Any antimorphism of J' must map the P_4 to itself (once again, this can be proved using Lemma 1.B) and J onto itself; so it too will contain cycles of unequal lengths. In this manner we obtain an infinite family of counterexamples for each feasible n:

4. Proposition. *There is a sc-graph on n vertices whose antimorphisms all have non-trivial cycles of unequal length (specifically, at least one cycle of length 4 and at least one of length 8) if and only if $n \equiv 0$ or $1 \pmod 4$, $n \geq 12$.* □

Apart from Hartsfield's graph, the graphs we constructed are not regular, so they are not counterexamples to Kotzig's problem. With a little work, however, we can use Hartsfield's graph as the basis for an infinite family of counterexamples on $n = 13 + 8k$ vertices. Our construction is loosely based on one by Colbourn and Colbourn [7].

For any natural number k, we construct $H(k)$ as follows (see Figure 3 for a sketch). We add two K_{2k}'s and two $\overline{K}_{2k}$'s. In each of these four graphs, the first k vertices are joined to every vertex of N, the last k vertices are joined to every vertex of $\overline{N}$. Every

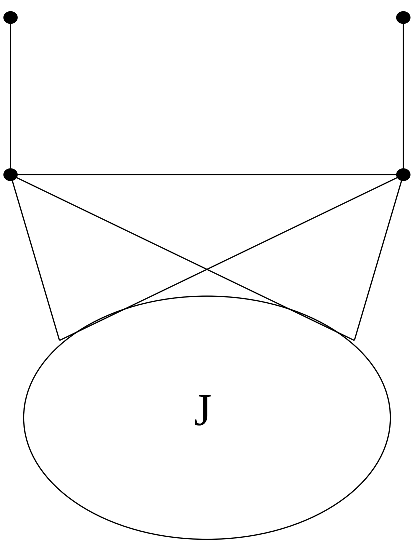

Figure 2: New counterexamples from old

vertex of the K_{2k}'s is also joined to the vertex 1. Finally, every vertex of each $\overline{K}_{2k}$ is joined to every vertex of the other $\overline{K}_{2k}$, and every vertex of one of the K_{2k}'s.

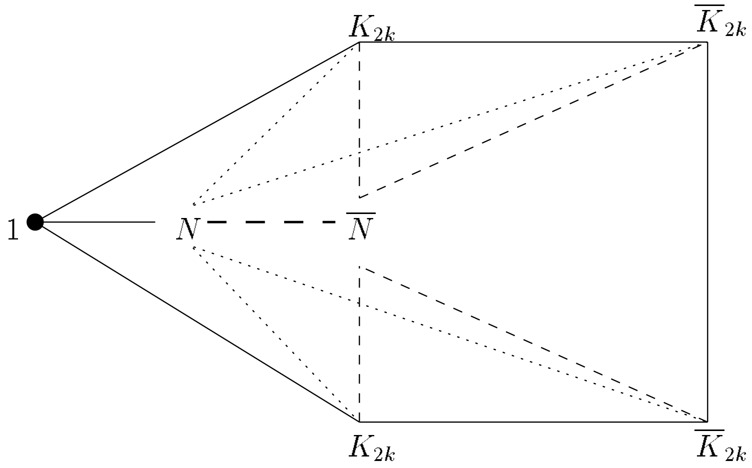

Figure 3: Regular sc-graphs with antimorphisms of unequal cycle lengths

This graph has $8k+13$ vertices, is regular of degree $4k+6$ and self-complementary. We want to show that any antimorphism of $H(k)$ must map H to itself, and thus must induce one of the antimorphisms of H mentioned above, meaning that it will contain cycles of length 1, 4 and 8.

To do this we note that if a vertex is in x cliques (maximal complete subgraphs) of size y, then an antimorphism must map it to a vertex that is in x maximal independent sets of size y. The next table lists the number of cliques and maximal independent sets containing each vertex of $H(k)$; we use y^x to mean that there are x cliques (or maximal independent sets) of size y; verifying the details in the table is routine but tedious. We can now see that the vertices of N must map onto vertices of $\overline{N}$ and vice versa (for $k > 3$ this is easy to see, as they are the only vertices not contained in any clique or maximal independent set of size $2k+1$; for $k \leq 3$ a little more checking is needed, but this is quickly accomplished). We can check similarly that the vertex 1 must map to itself.

5. Definition. An *almost-regular graph* is a graph with exactly two degrees, s and $s+1$, for some s. For any vertex v in a regular graph G, $G-v$ is almost-regular.

We noted previously that $H-1$ is an almost-regular sc-graph whose antimorphisms all have cycles of unequal length; the same reasoning is valid for $H(k)-1$. So we have the following:

vertices in:	cliques	max.ind.sets
$\{1\}$	$(k+4)^4, (2k+1)^2$	$(k+4)^4, (2k+1)^2$
first k vertices of K_{2k}	$(k+4)^{2k+2}, (2k+1)^{2k+1}$	$2^k, 5^{2k}, (k+4)^2, 2k+1$
first k vertices of $\overline{K}_{2k}$	$2^k, 5^{2k}, (k+4)^2, 2k+1$	$(k+4)^{2k+2}, (2k+1)^{2k+1}$
last k vertices of K_{2k}	$(k+3)^{9k}, (2k+1)^{2k+1}$	$2^k, 3^{9k}, (k+3)^9, 2k+1$
last k vertices of $\overline{K}_{2k}$	$2^k, 3^{9k}, (k+3)^9, 2k+1$	$(k+3)^{9k}, (2k+1)^{2k+1}$
N	3^2 or $3^3, 4, 5^{k^2}, (k+4)^{2k+2}$	either $3^4, 4^{3k^2}, (k+3)^{6k}$ or $3^2, 4^{3k^2+1}, (k+3)^{6k}$
$\overline{N}$	either $3^4, 4^{3k^2}, (k+3)^{6k}$ or $3^2, 4^{3k^2+1}, (k+3)^{6k}$	3^2 or $3^3, 4, 5^{k^2}, (k+4)^{2k+2}$

We have thus obtained, for all integers $n = 13+8k$ [respectively, $n = 12+8k$], $k \geq 1$, a regular [respectively, almost-regular] sc-graph whose antimorphisms all contain non-trivial cycles of unequal length, specifically at least one cycle of length 4 and at least one of length 8.

3 Counterexamples to Kotzig's second conjecture

It is clear from the definitions in Section 1 that $F(G) \subseteq \hat{F}(G)$. Kotzig [16] asked whether equality holds for all regular sc-graphs, but Rao [31] found a graph G_9 on 9 vertices for which the inclusion is strict. He claimed to have an infinite family of such examples, but the last two authors [27, 28] pointed out that his construction works for G_9 only. However, they found two other graphs G_{17} and G_{33} on 17 and 33 vertices, respectively, with the desired property. All three graphs actually satisfy the stronger property

$$F(G) \subsetneqq V(G) = \hat{F}(G) \qquad (*)$$

By Thm. 8.I, J and Thm. 9.A, B below, this is equivalent to saying that they are strongly vertex triangle regular (SVTR) but not vertex-transitive. Nair and Vijayakumar [28] showed that the lexicographic product[1] $G(H)$ of two SVTR graphs will also be SVTR; it can also be checked that $G(H)$ is self-complementary if G and H are both self-complementary. Moreover, Imrich [14, Thm. 9.4] proved that $G(H)$ is vertex-transitive if and only if G and H are both vertex-transitive. Finally, any vertex-transitive graph is trivially SVTR.

Thus the lexicographic product of any (positive) number of copies of G_9, G_{17} and G_{33}, and any vertex-transitive sc-graph, also satisfies $(*)$. So (using the last part of Thm. 9) we can construct such graphs on n vertices, for each n of the form $9^\alpha 17^\beta 33^\gamma N$, where N is a sum of two squares and α, β, γ are non-negative integers, not all zero.

4 Self-complementary digraphs

Problems 3.B, C and D all ask for information about the sets $F(G)$ and $N(G)$. In an important paper, Rao [31] characterised these sets and also gave significant details on the structure of the orbits of G (i.e. equivalence classes under the automorphism group).

A key part of his characterisation of $F(G)$ was to show that it forms an orbit. The proof, first given by Robinson [33], depended on the structure of antimorphisms of sc-graphs, and did not carry over to sc-digraphs. Eplett [8] proved a similar result for sc-tournaments using properties of Sylow groups, but this did not extend to general sc-digraphs either. However, Robinson gave another proof that *did* apply to sc-digraphs, producing the following result [34]:

6. Theorem. *For a self-complementary digraph D, the set $F(D)$ forms an orbit under $\mathcal{A}(D)$. All antimorphisms permute the orbits of D, and $F(D)$ is the unique orbit which is fixed by any (and every) antimorphism.* $\square$

We note that Rao's results establish the equivalence of statements 9.A, 9.B and 9.C (see next section) for sc-graphs. Robinson's theorem generalises the equivalence of 9.A and 9.B to sc-digraphs, while in the next theorem we do the same for statements 9.B and 9.C.

For $X, Y \subset V(D)$ we use $D[X]$ to denote the subgraph induced by X, $D[X,Y]$ the bipartite subgraph induced by X and Y (with vertex-set $X \cup Y$, and edges $\{uv | u \in X, v \in Y\}$), and $E[X]$ and $E[X,Y]$ the corresponding edge-sets.

7. Theorem. *Let D be a self-complementary digraph. Then the orbits of D can be numbered $V_0, V_1, V_2, \ldots, V_{2s}$ such that:*

A. $F(D) = V_0$;

B. $\sigma(V_0) = V_0$ and $\sigma(V_i) = V_{2s+1-i}$ for any antimorphism σ;

C. $|V_0|$ is odd if $|V(D)|$ is odd, otherwise $V_0 = \emptyset$;

[1]The lexicographic product $G(H)$ is produced by replacing each vertex of G by a copy of H, and replacing each edge of G by all $|V(H)|^2$ possible edges between the corresponding copies of H.

D. $D[V_0]$ *is a regular sc-subgraph (of degree* $(|V_0|-1)/2$*, for* $V_0 \neq \emptyset$*), and* $D[V_i, V_{2s+1-i}]$ *is a regular bipartite self-complementary subgraph*[2] *(of degree* $|V_i|/2$*) for all* $i \geq 1$*.*

E. $N(D) = E[V_0] \cup \bigcup_{i=1}^{s} E[V_i, V_{2s+1-i}]$.

Proof: Robinson's theorem tells us that $F(D)$ is an orbit (proving A), and that any antimorphism σ maps orbits onto orbits, but only maps $F(D)$ onto itself; since σ^2 is an automorphism, the mapping on orbits must be an involution, which proves B. This also proves that there are an even number of orbits apart from $F(D)$, and since $|V_i| = |V_{2s+1-i}|$, when $|V(D)|$ is odd we must have $|F(D)|$ odd too. The fact that $F(D)$ is empty whenever D has an even number of vertices follows from a result analogous to Theorem 2 ('Every antimorphism of a sc-digraph consists of cycles of even length and, possibly, one fixed vertex'), thus establishing part C. Part D also follows from B; we omit the details.

We now prove part E. By virtue of B, we have that $N(D)$ is a subset of the right hand side, and we only have to show equality. Let $uv \in E[V_0]$, let σ be an antimorphism which fixes u, and $f(u) = v$ for some automorphism f. Then $f\sigma$ is an antimorphism with $f\sigma(u) = v$, and thus $uv \in N(G)$. Now let uv be an edge with $u \in V_i$, $v \in V_{2s+1-i}$ for some $i \geq 1$; choose an arbitrary antimorphism σ, let $\sigma(u) = w$ and $g(w) = v$ for some automorphism g. Then $g\sigma$ is an antimorphism with $g\sigma(u) = v$, so $uv \in N(G)$. $\square$

5 Regular self-complementary graphs

Regular graphs often have a great deal of structure, and there are special classes of regular graphs (such as the strongly regular, distance regular or vertex-transitive) which present even more structure. In general these special classes overlap partially, but in the self-complementary case several classes completely contain other classes, forming two grand hierarchies (Figure 4). Much of the second hierarchy holds even for (di)graphs which are *not* self-complementary. This classification is the work of 29 authors [2, 4, 5, 6, 9, 11, 12, 15, 16, 17, 21, 25, 26, 27, 28, 30, 31, 34, 35, 36, 37, 38, 40, 41, 42, 43] which is gathered here for the first time.

In Section 1 we defined the sets $\hat{F}(G)$, $F(G)$, and $N(G)$, and the parameters $t(v)$ and $\bar{t}(v)$ of a vertex v. Also, for any edge e of a graph G, we define $t(e)$ to be the number of triangles of G containing e. If $t(e)$ is the same for all edges of G, we say that G is *edge triangle regular*; if, moreover, the graph is regular, then we say that it is *strongly edge triangle regular*, or just SETR for short. The terms *vertex triangle regular*, *strongly vertex triangle regular* and SVTR are defined analogously.

A graph which is both vertex- and edge-transitive is said to be symmetric, whereas a (di)graph which is vertex- and arc-transitive is said to be strongly symmetric. With the exception of $\vec{K}_2$, all connected arc-transitive graphs and digraphs are vertex-transitive, and thus strongly symmetric. The concept of edge-transitivity is not defined for digraphs, so the term "symmetric digraph" is sometimes used to describe strongly symmetric digraphs.

[2] A bipartite sc-graph G is a spanning subgraph of $K_{a,b}$ that is isomorphic to its bipartite complement $K_{a,b} - G$.

We need to define the class of $\mathcal{P}^*$-graphs. These have vertex-set $GF(p^r)$, where $p \equiv 3 \pmod 4$ and r is even; two vertices being adjacent iff their difference is in the set $\{\omega^j : j \equiv 0, 1 \pmod 4\}$, for some fixed primitive root ω of the field. It can be shown that this does not depend on the choice of primitive root, and that the only $\mathcal{P}^*$-graph that is also a Paley graph has order 9.

8. The First Hierarchy. *Let $G \neq \vec{K}_2$ be a self-complementary graph or digraph. Then the following conditions are equivalent*

A. G is edge-transitive

B. G is arc-transitive

C. G is symmetric

D. G is strongly symmetric

E. G is distance transitive

A symmetric sc-digraph D on n vertices exists if and only if $n = p^r$ for some odd prime p; and D is either the Paley tournament (whenever $n \equiv 3 \pmod 4$), or the Paley graph or $\mathcal{P}^*$-graph (whenever $n \equiv 1 \pmod 4$), or an exceptional graph on 23^2 vertices.

The Paley digraphs on a prime number p of vertices are the only circulant symmetric sc-digraphs; they are also the only symmetric sc-digraphs on p vertices.

For sc-graphs, either of conditions A, B, C, D, E imply the following equivalent conditions:

F. G is strongly regular

G. G is strongly edge triangle regular

H. G is distance regular

and each of these, in turn, implies the following equivalent conditions:

I. G is strongly vertex triangle regular.

J. G is regular and $\hat{F}(G) = V(G)$.

Strongly regular sc-graphs on n vertices exist when $n \equiv 1 \pmod 4$ is a prime power; they do not exist when n is not the sum of two squares. □

If p is a prime divisor We use $a^b||c$ to mean that a^b divides c, but a^{b+1} does not.

9. The Second Hierarchy. *If G is a self-complementary graph or digraph, then it is not a Graphical Regular Representation and the following are equivalent:*

A. G is vertex-transitive.

B. $F(G) = V(G)$.

C. $N(G) = E(G)$.

Infinite vertex-transitive sc-graphs exist if n is countably infinite or of order the power of the continuum. Finite vtsc-graphs exist if and only if $n = 4k + 1$ is the sum of two squares, that is, iff $p^r||n \Rightarrow p^r \equiv 1 \pmod 4$ (for p prime), while circulant sc-graphs exist iff $p|n \Rightarrow p \equiv 1 \pmod 4$ (for p prime).

Every circulant digraph is a self-converse Cayley digraph, which in turn is a vertex-transitive (or coset) digraph. For digraphs D on a prime number p of vertices this hierarchy collapses (that is, D is vertex-transitive iff it is circulant) and, moreover, consists just of graphs and tournaments (the vertex-transitive sc-graphs on p vertices, and all vertex-transitive tournaments). On p^2 vertices (where p is prime, $p \equiv 3 \pmod 4$), every circulant sc-digraph is a tournament.

Circulant sc-tournaments are just the circulant tournaments, which exist for all odd n.

The only symmetric circulant sc-digraphs are the Paley graphs and Paley tournaments on a prime number of vertices.

□

We note that one of the problems posed in the survey [9, 3.44.B(2)] has been solved by V. Liskovets [22] — mixed sc-circulants (i.e. circulant sc-digraphs which are neither graphs nor tournaments) do exist. One easy construction is to take the lexicographic product $X(Y)$ of two self-complementary circulants X and Y, so long as X and Y are not both tournaments and not both graphs. By taking X to be a circulant sc-graph on x vertices, and Y a circulant sc-tournament on y vertices, we obtain mixed sc-circulants on xy vertices, for any x with prime divisors all congruent to $1 \pmod 4$, and any odd y.

Li and Praeger [20] constructed infinitely many vertex-transitive self-complementary graphs that are not Cayley graphs; while Liskovets and Pöschel [23, 24] and, independently, Li and Jajcay [19] constructed infinitely many circulant sc-graphs and sc-digraphs with no Cayley isomorphism to their complement.

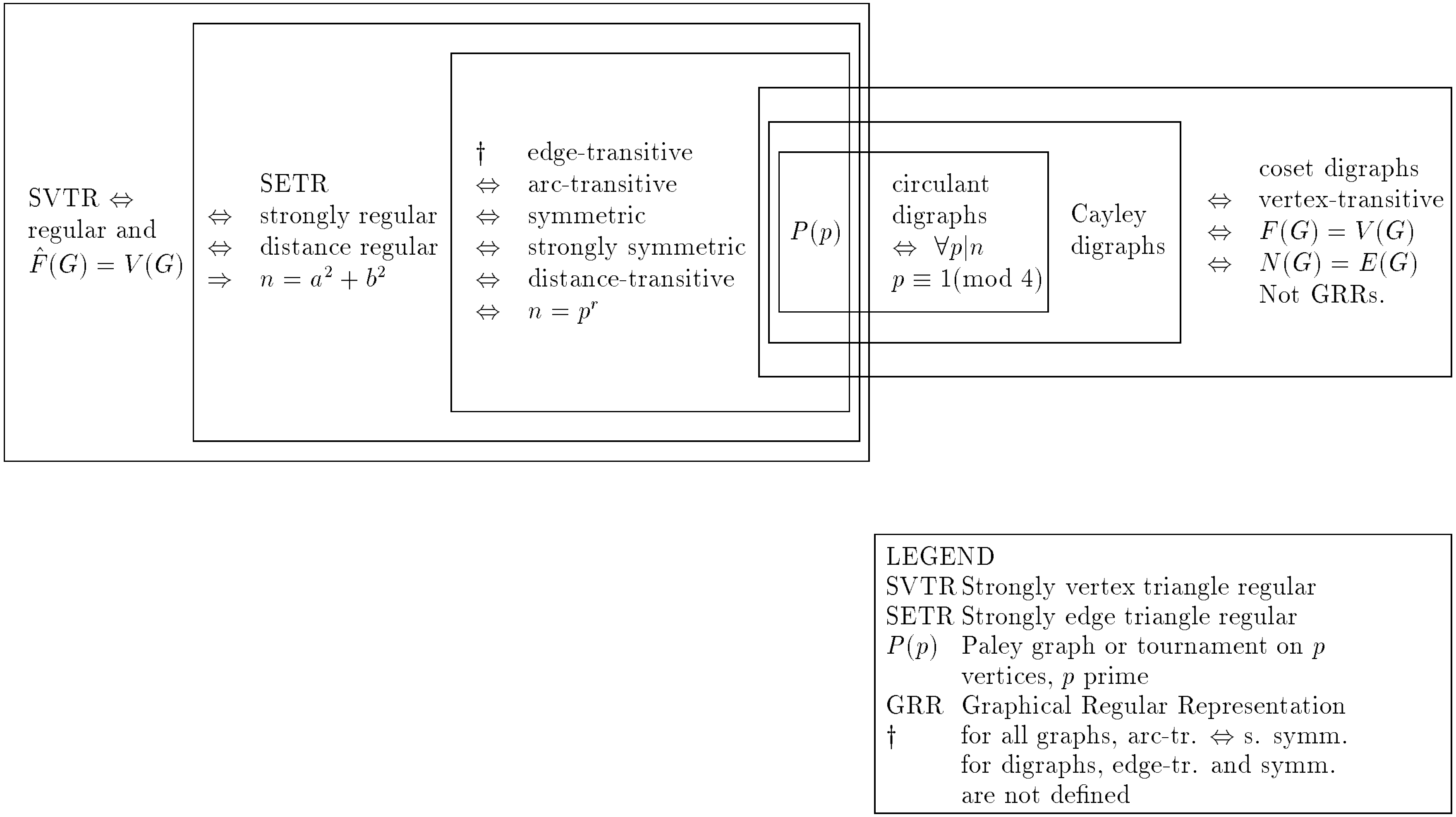

Figure 4: Regular self-complementary graphs: two interlocking hierarchies

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