Self-complementary graphs and generalisations: a comprehensive reference manual

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August 1999

This work was undertaken in part fulfilment of the Masters of Science degree at the University of Malta, under the supervision of Prof. Stanley Fiorini.

Abstract

A graph which is isomorphic to its complement is said to be a self-complementary graph, or sc-graph for short. These graphs have a high degree of structure, and yet they are far from trivial. Suffice to say that the problem of recognising self-complementary graphs, and the problem of checking two sc-graphs for isomorphism, are both equivalent to the graph isomorphism problem.

We take a look at this and several other results discovered by the hundreds of mathematicians who studied self-complementary graphs in the four decades since the seminal papers of Sachs (Über selbstkomplementäre graphen, *Publ. Math. Drecen* **9** (1962) 270–288. MR 27:1934), Ringel (Selbstkomplementäre Graphen, *Arch. Math.* **14** (1963) 354–358. MR 25:22) and Read (On the number of self-complementary graphs and digraphs, *J. Lond. Math. Soc.* **38** (1963) 99–104. MR 26:4339).

The areas covered include distance, connectivity, eigenvalues and colouring problems in Chapter 1; circuits (especially triangles and Hamiltonicity) and Ramsey numbers in Chapter 2; regular self-complementary graphs and Kotzig's conjectures in Chapter 3; the isomorphism problem, the reconstruction conjecture and self-complement indexes in Chapter 4; self-complementary and self-converse digraphs, multi-partite sc-graphs and almost sc-graphs in Chapter 5; degree sequences in Chapter 6, and enumeration in Chapter 7.

This is a manual more than a survey, as it contains all the results I could find, and quite a few proofs. There are also a few original results. For any queries, comments or suggestions, please contact me at *afarrugia at alumni dot uwaterloo dot ca*.

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Acknowledgements

0.1. Writing this thesis has been like working on a four-hundred piece jigsaw puzzle, except that instead of houses and trees, the pieces contained theorems and equations, sometimes in different languages and very different notation, not to mention that they were (literally) scattered around the globe. For helping me to gather some of the pieces, I must thank all those who generously sent reprints of their papers (and sometimes even other people's papers):-

- Michael Albertson, Giorgio Balconi, Lowell Beineke, Paul Camion, Gary Chartrand, Charles Colbourn, Tamonas Gangopadhyay, Frédéric Havet, A.J.W. Hilton, Bill Kocay, Jan Kratochvíl, Flavia Mamanna, Terry McKee, Ronald Read, Chris Rodger, Alexander Rosa, Ivo Rosenberg, Norma Salvi-Zagaglia, Allen Schwenk, A. Vijayakumar, Earl Glen Whitehead, Jr. and A. Pawel Wojda;

- Stanley Fiorini who obtained other papers, and lent me the invaluable four-volume Graph Theory Reviews and other books; Irene Sciriha, who brought to my attention Stockmeyer's papers on the reconstruction conjecture and lent me various books; and Josef Lauri who gave me photocopies of other papers.

- Gino Favero who sent me photocopies of over 50 papers which were not available in Malta; and Susan Raimondo and Daniele Varacca, who sent other photocopies;

- the journals section of the University of Malta Library, whose value is sometimes questioned by people who have never done any research.

I would also like to thank the following people for their help:

- my employers, Magus Int.l, and especially Jackie Chetcuti, Mario Borg and Arnold Stellini, for granting and then extending more than a week's leave at point blank notice, so that I could finish my thesis

- Peter Wiosna, who translated parts of H. Sachs' "Über selbstkomplementäre graphen", the first paper on self-complementary graphs;

- those who provided information: Edward Bender, Luigi Borzacchini, Gary Chartrand, Christopher Clapham, Gouli Ding, Luis Goddyn, Bill

Kocay, Brendan McKay, Rhys Price Jones, Erich Prisner, Mukesh Punhani, Igor Rivin, Ivo Rosenberg, Allen Schwenk, Jozef Širáň and Detlef Wille;

- those who helped me get in touch with other people: Anne Bres (secretary at the Institut National de Recherche en Informatique et en Automatique), Dimitri Chernyak, Vladimir Chernyak, Mike J. Grannell, Terry Griggs, Fausta Guzzoni (secretary at the University of Parma), Jo Ann McFarland (secretary at the University of California), Steve Quinn, Catherine P. Vistro-Yu, C.R. Rao, John Sheehan and Robin J. Wilson;

- and staff at the Department of Maths for helping in various ways — James L. Borg, who provided lots of technical assistance; Joseph Muscat and the secretary, Annabel Attard.

I am very grateful to Brian Alspach for suggesting Juraj Bosák's "Decompositions of graphs" when I was still starting on my research. It is an excellent survey book which I have often referred to both for content and style; and from which I first learnt that there is more to self-complementary graphs than the "$n \equiv 0$ or $1 \pmod 4$" lemma.

I thank Prof. Fiorini for his patience and help; and my parents for their support. Last, but definitely not least, I would like to thank my sister Nadine who drew and re-drew (and re-drew[1]) the diagrams.

Updates and Correspondence

0.2. I intend to publish this survey on the internet and to keep it updated as far as possible. If you have any comments, suggestions, corrections, queries or new results, please feel free to contact me at

Notation and references

0.3. We assume a basic knowledge of graph theory, and refer to Wilson [387] for any definitions and notation not given here. Unless stated otherwise,

[1]If only some brothers were less perfectionist!

all the results apply to finite (self-complementary) graphs without loops or multiple edges. In particular, we use $v \sim w$ to denote that two vertices v and w are adjacent. We often use $|E(G)|$ for the number of edges of G, as m or $m(G)$ may be used for different purposes. For any subset X of $V(G)$, we denote by $G[X]$ the subgraph induced by X, and $E[X]$ the edge-set of $G[X]$. For disjoint subsets X, Y, we use $G[X, Y]$ to denote the bipartite subgraph with vertex set $X \cup Y$ and edge set $E[X, Y] = \{uv \in E(G) | u \in X, v \in Y\}$. We define a clique to be a complete subgraph that is not contained in a larger complete subgraph. We use $\Box$ to denote that a proof has ended, or that no proof will be given.

We refer to publications either by their bibliographical number, e.g. [341], or, in Theorems, by author and year, e.g. [Sachs 1962]. We sometimes state two analogous definitions simultaneously, using square brackets for clarity, e.g. "A signed [marked] graph is a graph with a + or − sign assigned to each edge [vertex]."

0.4. Where there is no ambiguity, we abbreviate "self-complementary graph" or "self-converse graph" to *sc-graph*. We could have used "s.c. graph" or "s-c graph", but the chosen notation, due to Rosenberg [331], is useful because it can be easily extended to related concepts, as in Table 1. Some of the

Type of graph	Abbreviation
self-complementary graph	sc-graph
regular self-complementary graph	rsc-graph
strongly regular self-complementary graph	srsc-graph
vertex transitive self-complementary graph	vtsc-graph
almost self-complementary graph	asc-graph
bipartite self-complementary graph	bipsc-graph
tripartite self-complementary graph	tripsc-graph
r-partite self-complementary graph	r-psc-graph
t-complementary graph	t-c-graph

Table 1: Abbreviations used.

notation is adapted from that of Gangopadhyay (c.f. [135]). In the case of tournaments we almost always use the abbreviated form, sc-tournaments, to underline the fact that a self-complementary tournament is the same thing as a self-converse tournament.

We note that "bipartite self-complementary graph" is defined as in 5.13. We *never* take it to mean "a self-complementary graph which happens to be bipartite" — this is not an interesting concept as there is only one such graph, namely P_4. Similar considerations hold for tripartite and r-partite self-complementary graphs.

Although isomorphisms from a graph to its complement are normally called complementing permutations, there are other names for them — Kotzig [227] called them isomorphism permutations, while other authors (e.g. Eplett [112], P.S. Nair [266], Balconi and Torre [31]) call them anti-automorphisms. We use *antimorphism*, which is both meaningful and easy to pronounce. Theorem 1.30 gives added significance to this terminology.

Layout of the thesis

0.5. Chapter 1 describes the fundamental structural properties of sc-graphs and their antimorphisms; it also contains several miscellaneous results which do not warrant a chapter to themselves. Chapter 2 covers results on paths, circuits and cliques in self-complementary graphs, while Chapter 3 is about regular sc-graphs and the very strong properties they enjoy (or suffer from).

Chapter 4, Self-Complementarity, discusses the problem of generating self-complementary graphs, and distinguishing them from each other and from non-self-complementary graphs. It also shows how sc-graphs have been either used as tools, or investigated in their own right, in such areas as the isomorphism problem, the reconstruction conjecture, codes and information. Chapter 5 describes several related concepts such as multipartite self-complementary graphs and almost self-complementary graphs.

The final two chapters are meant as a useful reference guide to results on degree sequences and enumeration of sc-graphs. Virtually no proofs are presented in these two chapters, but the enumeration results are useful even to those not particularly interested in enumeration, because they show that many self-complementary structures described in Chapter 5 are linked not only by an intuitive resemblance, but also by very similar counting formulas.

0.6. All the work here is attributed to its original author (or authors, if it was discovered more than once), except for some simple results which are known and used long before anyone bothers to state them formally. There is also some original work:

- we showed that any sc-graph on $n \geq 4$ vertices is contained in a sc-graph on $n+4$ vertices with diameter 3 but no end-vertices (1.28)

- we point out that known results imply a partial generalisation of Turán's theorem, namely that every graph on $n \geq 6$ vertices with $m \geq \frac{n(n-1)}{4}$ edges must contain a C_3, C_4, C_5 or C_6 (2.2)

- we pointed out certain difficulties in a construction of Nair and Vijayakumar, and resolved them after some correspondence with these two authors (3.40)

- we used a graph found by Hartsfield [198] to construct an infinite family of sc-graphs whose antimorphisms have unequal cycle lengths (3.45)

- we characterised disconnected bipsc-graphs with isolated vertices (5.24)

- we obtained a simpler proof of a result by Gangopadhyay and Rao Hebbare [138] on the diameter of connected bipsc-graphs, also obtaining a result on the radius of these graphs (5.28)

We also simplified or extended certain results:

- we simplified a proof by Wojda and Zwonek [391], pointing out that it implies a known result on circulant graphs (1.70)

- two proofs by Colbourn and Colbourn [93, 94] are slightly simplified (4.8, 4.10)

- we showed that an important theorem by Rao [306] on sc-graphs can be extended to sc-digraphs by making use of a result by Robinson [324] (5.7)

- Sachs' [341] and Ringel's [320] theorem on the structure of antimorphisms of sc-graphs (that is, cyclic 2-morphisms) is extended to cyclic t-morphisms, thus obtaining an existence result for cyclically t-complementary graphs, and showing up an error in two published theorems on t-complementary graphs (5.53)

What was left out, and what wasn't

0.7. What are we referring to when we mention self-complementary structures? There are many ways of generalising self-complementary graphs, giving a spectrum of related concepts which stretches out until we have lost the "feel" of dealing with anything related to sc-graphs. A line has to be drawn somewhere.

The first generalisation is that of going from graphs to multigraphs, digraphs, relations (digraphs with loops), hypergraphs and so on. All of these have self-complementary versions defined in the obvious way, and the similarities are obvious and close.

0.8. The next generalisation concerns the concept of complement itself. The complement of a graph G can be defined more precisely as $\overline{G} = K_n - G$, where $|V(G)| = n$. We can call this the complement of G with respect to K_n. In general, for any graph H, and any subgraph $G \subseteq H$ with $V(G) = V(H)$, we can define the complement of G with respect to H as $\overline{G} = H - G$. When H is a complete or almost complete graph (e.g. a complete digraph, a complete multi-partite graph, or $K_n - e$), and $G \cong H - G$, the resemblance to sc-graphs is quite obvious; we deal with these graphs in 5.13 – 5.34. However, an important feature has already been lost — the generalised complement is not always unique, that is, we could have $G_1 \cong G_2$ but $H - G_1 \ncong H - G_2$, even when H is, say, a complete bipartite graph. We do not consider at all the case where H is not complete or almost complete.

0.9. Self-complementary graphs are interesting not only because of their links with other areas of graph theory (outlined in Chapter 4) but also because they form an infinite, and yet scarce, class of graphs, and have strong structural properties. For example, the classical self-complementary graphs must have exactly $\frac{1}{2}\binom{n}{2}$ edges and diameter 2 or 3; they exist for every feasible value of n, and yet the proportion of sc-graphs to graphs with the same number of vertices (and even with the same number of edges), tends to 0 (see 7.15).

Some authors (e.g. Nara [267], Gangopadhyay [141, 145]) considered what they called self-placeable or self-packing graphs, that is, graphs isomorphic to a subgraph of their complement. While this class of graphs includes all of the concepts described above as special cases, we consider it to be too far from the spirit of sc-graphs, and also too large and amorphous (for example, it is known that all graphs with n vertices and at most $n-2$ edges are self-packing).

0.10. A further generalisation is to look at self-complementary graphs as factorisations of K_n into two isomorphic subgraphs, and then consider what happens when we factorise K_n (or some other graph) into any number of isomorphic subgraphs. Isomorphic factorisations of graphs have been extensively studied, and, if only for this reason, our overview of the area (5.45 – 5.59) is rather compact. To do justice to it would require another thesis or two.

0.11. The concept of self-dual graphs includes most of the generalisations which, to my mind, remains close to the spirit of sc-graphs. By "self-dual" we mean graphs satisfying $G \cong X(G)$, where

- X is an operator defined unambiguously on some class $\mathcal{C}$ of graphs, and
- X is symmetric, that is $X(X(G)) = G$ for all graphs G in $\mathcal{C}$ (note that this implies that X is a bijection on $\mathcal{C}$).

One familiar case is the operation of reversing all arcs in a digraph; self-converse digraphs bear some similarity to self-complementary digraphs, and coincide with them in the case of tournaments; we consider them in many places, but especially 5.2–5.12.

Another example concerns the reversal of all signs on signed and marked graphs (graphs with a $+$ or $-$ sign on the edges or vertices, respectively); in fact, self-dual complete signed graphs are essentially the self-complementary graphs. We consider these concepts in 5.60–5.63 and 7.38–7.54 but, as with isomorphic factorisations, the potential number of self-dualities makes it impossible to cover the area in any depth.

We originally thought (and hoped!) that topologically self-dual maps had little, if any, connection with self-complementary graphs, but a few days before finishing the thesis we came across an interesting paper by White [382], on which we report briefly in 3.25.

The line graph $L(G)$ and total graph $T(G)$ are defined unambiguously, but we rarely have $L(L(G)) = G$ or $T(T(G)) = G$. Like the topological dual, they do not even preserve the number of vertices. These operations are distinctly unlike complementation and, except for a short mention in 1.55 we do not include them in our survey.

Many authors have even considered graph equations which are even further from the spirit of self-complementary graphs, e.g. $L(G) = T(G)$ or $L(G) = T(H)$ for some graphs G, H. For such matters we refer the reader to Cvetković and Simić [100] and Prisner [290].

0.12. Regretfully, I have to include several diagrams among the list of things left out. The software I am using, LaTeX, is wonderful for mathematical notation and cross-references (of which there are quite a few in this thesis) but notoriously troublesome when it comes to even the simplest of diagrams. At many points, the reader will have to use the trusted method of pencil and paper; I hope that the proofs will serve well as instructions on how to sketch the diagrams.

A word of warning

0.13. When surveying the work of hundreds of mathematicians it would be a brave person who claims to understand it all, and there is one area where I must admit to being somewhat shaky, namely the vertex-transitive self-complementary digraphs of prime order, their automorphism groups and enumeration (3.21–3.22, 3.30–3.31, and 7.18–7.25). I have tried to report the results on these topics as accurately as possible, but cannot give a complete guarantee of correctness.

Chapter 1

Introduction and Fundamental Properties

1.1. The study of self-complementary graphs was initiated by Sachs' impressive 1962 paper [341] and (later, but independently) by Ringel [320]. The subject had an auspicious birth, and its development over the last four decades was charted by Rao [302]and Bosák [45, Chapter 14]. There have also been four Ph.D. theses devoted to it at least in part [63, 134, 150, 263], and four others which cover related topics [201, 293, 337, 383], while the number of papers now runs into the hundreds. A new survey is needed, and that is the purpose of this thesis. Our aim is not just to provide pointers to the considerable amount of research that has been done, but also to give explicit results. Where possible we give proofs as well to whet the reader's appetite.

1.2. The complement $\overline{G}$ of a graph G has the same vertices as G, and every pair of vertices are joined by an edge in $\overline{G}$ if and only if they are not joined in G. A self-complementary graph G is one that is isomorphic to its complement $\overline{G}$. It is important to realise that, although G and $\overline{G}$ are isomorphic graphs with the same set of vertices, they are nonetheless distinct; in fact, they have no edge in common. Thus G and $\overline{G}$ will have the same properties, but any given vertex or set of vertices will generally have different properties in G and $\overline{G}$.

We start with the most basic result on self-complementary graphs, one included even in introductory courses on graph theory.

Lemma. *If G is a self-complementary graph on n vertices, then $|E(G)| = \frac{n(n-1)}{4}$, and $n \equiv 0$ or $1 \pmod 4)$.*

Proof: Since $G \cup \overline{G} = K_n$, and $|E(K_n)| = \binom{n}{2}$, we must have $|E(G)| = |E(\overline{G})| = \frac{1}{2}\binom{n}{2} = \frac{n(n-1)}{4}$. Moreover, this must be an integer, and since we cannot have $2 \mid n$ and $2 \mid n-1$, we must either have $4 \mid n$ or $4 \mid n-1$. $\square$

1.3. This result can be used as a quick check — for example, Sridharan defines cad and polycad graphs in [359], and proves that these classes of graphs have $n+2$ and $2n+3$ edges respectively; his result that these graphs are not self-complementary then follows easily.

The lemma is also useful in finding small sc-graphs — for n at most 7, there can only be self-complementary graphs on 1, 4 or 5 vertices, with 0, 3 or 5 edges respectively; in fact there are just four, which are shown in Figure 1.1. They are known as K_1 (also called the trivial sc-graph), P_4, C_5, and the A-graph or bull-graph. Other self-complementary graphs were catalogued in [22, 375] ($n = 8$), [256, 257] ($n = 8$ and 9), [232] ($n = 12$), [118] ($n \leq 12$) and [252] ($n = 13$).

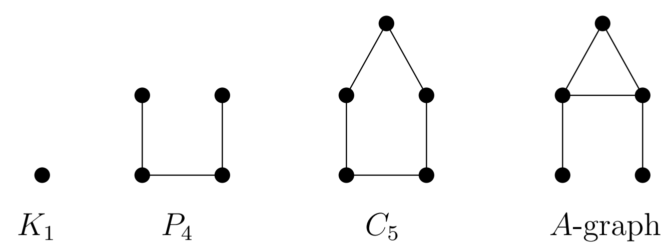


Figure 1.1: The small sc-graphs

For the sake of clarity, in what follows we will often denote the number of vertices in a sc-graph by $4k$ or $4k+1$.

At various points we will also mention or make use of bipartite self-complementary graphs with respect to $K_{m,n}$ — graphs $G \subset K_{m,n}$ such that $G \cong K_{m,n} - G$. We tackle them in more detail in Chapter 5.

Distance and connectivity

1.4. We now turn to some simple but powerful results concerning distance in self-complementary graphs, starting with some definitions.

The distance between two vertices v and w, denoted by $d(v,w)$, is the length of a shortest path between them, or ∞ if there is no such path. Thus $d(v,v) = 0$, while $d(v,w) = 1$ if and only if v and w are adjacent.

The eccentricity of a vertex v is the maximum of all distances $d(v,w)$. The diameter [radius] of a graph G is the maximum [minimum] of the eccentricities of all vertices of G, and is denoted by $diam(G)[rad(G)]$. Thus G is disconnected if and only if $diam(G) = rad(G) = \infty$; connected graphs have finite radius and diameter.

An edge vw of a graph G is a dominating edge of G if all vertices of G are adjacent to either v or w, or both.

Lemma. *A vertex v has eccentricity at least 3 in G if and only if, in $\overline{G}$, it lies on a dominating edge and has eccentricity at most 2.*

Proof: If v has eccentricity at least 3 in G, then there is some other vertex w such that $d(v,w) \geq 3$. Thus v and w are not adjacent, and there is no vertex adjacent to both of them. Then in $\overline{G}$, v and w will be adjacent; and every other vertex will be adjacent to either v or w, or both. Thus vw is a dominating edge of $\overline{G}$, and v has eccentricity at most 2. It is easy to see that the converse holds too.

1.5. Corollary. *For any graph G the following hold:*

A. *If $rad(G) \geq 3$ then $rad(\overline{G}) \leq 2$.*

B. *$Diam(G) \geq 3$ if and only if $\overline{G}$ has a dominating edge.*

C. *If $diam(G) \geq 3$ then $diam(\overline{G}) \leq 3$.*

D. *If $diam(G) \geq 4$ then $diam(\overline{G}) \leq 2$.*

Proof: If $rad(G) \geq 3$ then every vertex has eccentricity at least 3, so that in $\overline{G}$ every vertex will have eccentricity at most 2.

As for B and C, $diam(G) \geq 3$ if and only if there is some vertex v of eccentricity at least 3 in G, if and only if $\overline{G}$ contains a dominating edge.

Moreover, it is quickly checked that a graph with a dominating edge has diameter at most 3 (though the converse is not true, e.g. C_5). Further, if $diam(G) \geq 4$ then $diam(\overline{G}) \neq 3$, as otherwise $diam(G) = diam(\overline{\overline{G}}) \leq 3$. $\square$

1.6. Theorem. *Let G be a non-trivial self-complementary graph; then*

A. G has radius 2 and diameter 2 or 3.

B. G has diameter 3 if and only if it contains a dominating edge.

C. The number of vertices of eccentricity 3 is never greater than the number of vertices of eccentricity 2.

Proof: We claim that G has no vertices of eccentricity 1, that is vertices of degree $n-1$. For if it did, the complement would contain isolated vertices, and then every vertex would have eccentricity ∞ in $\overline{G}$. Since $G \cong \overline{G}$, this is impossible. Thus the radius and the diameter must both be at least 2; A and B then follow from 1.5.

Therefore G has only vertices of eccentricity 2 or 3. If there are t vertices of eccentricity 3, these will become t vertices of eccentricity 2 in $\overline{G}$, so that C is proved. $\square$

1.7. We note that 1.6.A was first stated by Ringel [320], but the proof given here is due to Harary and Robinson [190]. The results related to dominating edges, such as 1.6.B, are essentially due to Akiyama and Ando [6], while 1.6.C is a special case of a theorem of Tserepanov [372] on complementary graphs. A direct proof of 1.5.D was given in [365].

These simple but powerful results suggest that it might be difficult to find sc-graphs, maybe even that there is only a finite number of them. We will soon see that this is far from the truth.

1.8. The well-known result that G and $\overline{G}$ cannot both be disconnected is a corollary of 1.5; all self-complementary graphs are thus connected. We now look at how well connected they are, a question thoroughly researched by Akiyama and Harary [12], to whom the next few results are due.

A vertex v in a connected graph G is called a cut-vertex if $G - v$ is disconnected. A connected graph is said to be k-connected if the removal of less than k vertices leaves a subgraph that is still connected. Thus, if G has cut-vertices it is only 1-connected; if it does not have cut-vertices, it is (at least) 2-connected and is called a block. A vertex of degree one is called an end-vertex, and the number of end-vertices in G will be denoted by $v_1(G)$.

Lemma. *A self-complementary graph G has cut-vertices if and only if it has end-vertices.*

Proof: Since K_2 is not self-complementary, any sc-graph with end-vertices must have cut-vertices.

Now, if G is a sc-graph with cut-vertex v, but no end-vertices, then $G-v$ has at least two components. Let one of these components be A and let $G-v = A \cup B$. Then $\overline{G-v}$ contains a spanning bipartite subgraph, with parts A and B. Since G has no end-vertices, A and B each have cardinality at least two; and v has degree at least 2 in G, and thus also in $\overline{G}$. But then $\overline{G}$ is 2-connected, a contradiction. □

1.9. We therefore turn our attention to the presence of end-vertices in sc-graphs.

Lemma. *If a graph G has at least two end-vertices, then $\overline{G}$ has at most two end-vertices.*

Proof: Let v and w be two end-vertices of G, adjacent to x and y (possibly $x = y$). Then the only candidates for end-vertices in $\overline{G}$ are x and y, as all other vertices have degree at most $n-3$ in G. □

1.10. If H is a graph, we denote by $H + K_2 \circ K_1$ (Figure 1.2). the graph formed from H by adding four new vertices v_1, v_2, v_3 and v_4, the edges of the path $v_1v_2v_3v_4$, and the edges joining v_2 and v_3 with all the vertices of H. So the A-graph is $K_1 + K_2 \circ K_1$.

Theorem. *A graph G of order $n \geq 4$ has $v_1(G) = v_1(\overline{G}) = 2$ iff G is of the form $H + K_2 \circ K_1$, where H is a graph of order $n-4$.*

Proof: It is easily seen that $H + K_2 \circ K_1$ has the required property. Conversely, let $v_1(G) = v_1(\overline{G}) = 2$. Let v_1 and v_4 be the end vertices of G, adjacent to v_2 and v_3 respectively. As noted in the proof of Lemma 1.9, v_2 and v_3 are the only possible end-vertices in $\overline{G}$, so they must be distinct vertices, of degree $n-2$ in G. Then if we let H be the graph $G - \{v_1, v_2, v_3, v_4\}$, we

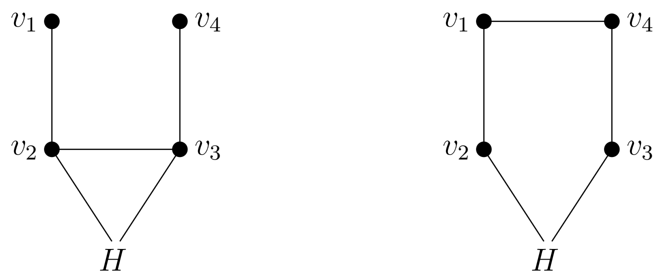


Figure 1.2: $H + K_2 \circ K_1$ and $H + \overline{K}_2 \cup P_4$

see that $G = H + K_2 \circ K_1$. □

1.11. Theorem. *For a sc-graph G of order n the following statements are equivalent:*

A. G has cut-vertices.

B. G has end-vertices.

C. $G = H + K_2 \circ K_1$, where H is a sc-graph of order $n-4$. In this case, G has exactly two cut-vertices, exactly two end-vertices, and diameter 3.

Proof: A $\Leftrightarrow$ B was proved in Lemma 1.8, and evidently C $\Rightarrow$ A, B. It is easy to check that in C, the graph G is self-complementary iff H is.

We now show that B $\Rightarrow$ C. By Lemma 1.9 a sc-graph G cannot have more than two end-vertices. We claim that it cannot have exactly one end-vertex. For, if G has just one end-vertex x, it must also contain one vertex y of degree $n-2$. In $\overline{G}$, x will be the unique vertex of degree $n-2$, and y the unique end-vertex. But x and y are adjacent in exactly one of G and $\overline{G}$, a contradiction. So, if G has end-vertices, it must have exactly two end-vertices, and C follows from Theorem 1.10. □

1.12. Let $\overline{G}_n$, $\overline{G}''_n$ and $\overline{G}^b_n$ be the number of non-isomorphic self-complementary graphs, self-complementary graphs with end-vertices, and self-complementary blocks on n vertices respectively. From 1.11, and from the fact that the mapping $H \to H + K_2 \circ K_1$ is one-to-one, we have the following result:

Corollary. *For any positive integer $n \geq 4$ we have*

$$\begin{aligned}\overline{G}_n &= \overline{G}''_n + \overline{G}^b_n \\ \overline{G}''_n &= \overline{G}_{n-4} \\ \overline{G}^b_n &= \overline{G}_n - \overline{G}_{n-4}.\Box\end{aligned}$$

1.13. The results just proved are significant in their own right, but they have a further important application. Starting from $G_1 = P_4$, and repeatedly performing the operation $G_{i+1} = G_i + K_2 \circ K_1$ we get an infinite family of self-complementary graphs of diameter 3 with $n \equiv 0 \pmod 4$). If we start with $G_1 = K_1$, we get a similar family for $n \equiv 1 \pmod 4$).

We thus know that sc-graphs (of diameter 3 and with end-vertices) exist for all feasible n. A simple modification will settle the existence question for sc-graphs of diameter 2. Let $H + \overline{K}_2 \cup P_4$ (Figure 1.2) denote the graph formed from H by adding four new vertices v_1, v_2, v_3 and v_4, the edges of the path $v_1v_2v_3v_4$, and the edges joining v_1 and v_4 with all the vertices of H. Then, starting with $G_0 = P_4$[1] or $G_1 = C_5$, we can form two new families of sc-graphs of diameter 2 for $n \equiv 0$ and $1 \pmod 4$). Incidentally, it can be checked that these families are Hamiltonian. In fact, every sc-graph has a Hamiltonian path (see 2.13), so $H + \overline{K}_2 \cup P_4$ is Hamiltonian for any choice of H.

1.14. Self-complementary graphs of diameter 3 without end-vertices can also be constructed easily (see Figure 1.3). We take a P_4, and an arbitrary graph H on k vertices. We then replace the end-vertices of the P_4 by copies of $\overline{H}$, and the interior vertices by copies of H. Where two vertices of P_4 were joined by an edge, the corresponding graphs are now joined by all possible edges between them. It is easy to see that the result is a self-complementary graph on $4k$ vertices with diameter 3, and for $k \geq 2$ there are no end-vertices. Moreover, if we add a vertex v, and join it to all the vertices of the copies of H, we get a sc-graph of order $4k+1$ and diameter 3 but without end-vertices. We state these results formally in the next theorem, of which A and B were first proved by Ringel [320].

1.15. Theorem.

[1] P_4 is not included in the sequence for $n \equiv 0 \pmod 4$ because its diameter is 3.

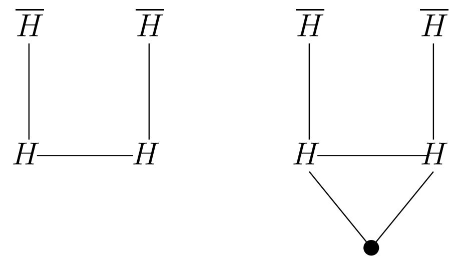


Figure 1.3: Sc-graphs of diameter 3 without end-vertices

A. *A (Hamiltonian) sc-graph of diameter* 2 *on* n *vertices exists iff* $n \equiv 0$ *or* $1 \pmod 4), n \geq 5$. *Every sc-graph on* n *vertices is an induced subgraph of a Hamiltonian sc-graph of diameter* 2 *on* $n+4$ *vertices.*

B. *A sc-graph of diameter* 3 *with end-vertices exists iff* $n \equiv 0$ *or* $1 \pmod 4$, $n \geq 4$. *Every sc-graph on* n *vertices is an induced subgraph of a sc-graph of diameter* 3 *and order* $n+4$ *with end-vertices.*

C. *A sc-graph of diameter* 3 *without end-vertices exists iff* $n \equiv 0$ *or* 1 $\pmod 4)$*)*, $n \geq 8$. □

1.16. We have thus seen how to produce systematically larger sc-graphs by adding a P_4; in 1.28 we will see how to construct sc-graphs of diameter 3 without end-vertices in this manner, thus filling an evident gap in part C of the previous theorem. In the case of sc-graphs with end-vertices the process is reversible. So when G is a graph with end-vertices, testing for self-complementarity quickly reduces to testing a subgraph H on $n-4$ vertices.

In view of the link between sc-graphs and the isomorphism problem explained in 4.2–4.10), it would be interesting to know whether there is a systematic way of producing smaller sc-graphs from larger ones, and whether this will let us test recursively for self-complementarity by considering smaller and smaller subgraphs.

In fact P_4's are plentiful in a self-complementary graph. Alavi, Liu and Wang [16, Lemma 1] have shown that every connected graph with connected complement must contain an induced P_4. When the graph is isomorphic to its complement we can say much more:

The Decomposition Theorem[Gibbs 1974]**.** *A self-complementary graph on $4k$ or $4k+1$ vertices contains k disjoint induced P_4's.* □

Unfortunately, to find the P_4's without a brute force search we not only need to know that the graph is self-complementary, but also to have an explicit isomorphism from G onto its complement (an antimorphism — see 1.22). Moreover, removing one of the P_4's will not necessarily give us a self-complementary subgraph on $n-4$ vertices. The theorem is thus of little, if any, help in finding an efficient test for self-complementarity.

Similar issues are tackled in 1.36–1.39, where we see how to produce sc-graphs of order $4k+1$ from those of order $4k$, and consider some cases where the process is reversible.

1.17. Let $\kappa(G)$, $\lambda(G)$ and $\delta(G)$ denote the vertex-connectivity, edge-connectivity and minimum degree of G, respectively. It is known that for graphs in general $\kappa(G) \leq \lambda(G) \leq \delta(G)$. If, in the construction of 1.14, we take H to be the complete graph on k vertices, we obtain a sc-graph with $\delta(G) = 2k-1$ and $\kappa(G) \leq k$, so the first inequality can be strict even for sc-graphs. However, Rao [302] conjectured, and N. Vijayaditya proved the following:

Theorem. *If G is a sc-graph, then $\lambda(G) = \delta(G)$.* □

1.18. Rao [302] also made the following conjecture. He noted that, using Menger's theorem, it could be shown that the Paley graphs (see 3.18) verify the conjecture when $4k+1$ is a prime power.

Conjecture. *For every natural number k there is a vertex-transitive self-complementary graph G on $4k+1$ vertices with $\kappa(G) = 2k$.*

1.19. A dominating set S is a set of vertices such that every vertex not in S is adjacent to some vertex in S. The domination number $\gamma(G)$ is the size of a smallest dominating set of G. No self-complementary graph can have $\gamma(G) = 1$. If the diameter of G is 3, then $\gamma(G) = 2$, by 1.6.B.

There are also arbitrarily large sc-graphs with diameter 2 and $\gamma(G) = 2$. The circuit C_5 is one example; the C_5-join of $(\overline{K}_k, K_k, K_1, K_k, \overline{K}_k)$ or $(K_k, \overline{K}_k, K_1, \overline{K}_k, K_k)$, defined in 1.26, are two other examples on $4k+1$ vertices. In all these cases there are edges of the graph which are not contained in any triangles; this is no co-incidence.

Theorem[Brigham and Dutton 1987]**.** *The following are equivalent for any graph G of order $n \geq 3$:*

A. every two vertices have a common neighbour

B. diam$(G) \leq 2$ and every edge of G is in a triangle

C. $\gamma(\overline{G}) \geq 3$. □

1.20. The vertices of a graph with minimum [maximum] eccentricity are called *central* [*diametral*] vertices. If the radius and diameter of a graph are equal then every vertex is both central and diametral, and the graph is said to be *self-centred*.

The antipodal [centre] graph of G, $A(G)$, has the same vertex-set as G, and two vertices are adjacent in $A(G)$ if and only if their distance in G is equal to the diameter [radius] of G.

Let G be a self-complementary graph. If its diameter is 2 then we have $A(G) = C(G) = \overline{G} \cong G$ and we say that G is *self-antipodal* and *self-central*. If its diameter is 3, G cannot be self-antipodal or self-central because $A(G) \subset \overline{G}$ and $C(G) \subset \overline{G}$. So a self-complementary graph is self-antipodal if and only if it is self-central if and only if it is self-centred.

A graph is said to be antipodal if it is the antipodal graph $A(H)$ of some graph H. But it is known [27] that a graph is antipodal if and only if it is the antipodal graph of its complement (this is also true for digraphs [213]). So a self-complementary graph is antipodal if and only if it is self-antipodal.

Interestingly, Hendry [206] showed that for sc-graphs of diameter 3, $A(G)$ is bipartite; while Acharya and Acharya [3] showed that a bipartite graph is self-antipodal if and only if it is a bipartite self-complementary graph of diameter 3 (see 5.13 for definitions).

Finally, we note the following — Nair [263, Thm. 2.18] showed that (up to addition or deletion of isolated vertices) every graph is the antipodal graph of some (Hamiltonian) graph of diameter 2. Rao [299] showed that every graph G is the centre graph of a (i) Hamiltonian, (ii) Eulerian, (iii) k-connected, (iv) k-chromatic (where $k = \chi(G)$), and (v) total graph.

Degree sequences and antimorphisms

1.21. We now look briefly at the degree sequences of sc-graphs, and the isomorphisms between a sc-graph and its complement. Many of the results in this section and the next are straightforward, and are not stated explicitly by any author. However, they are all useful, either as a quick check or an aid to intuition, or in proving other results.

The degree of a vertex is the number of vertices to which it is adjacent. The degree sequence of a graph is the sequence of degrees of all its vertices arranged in non-increasing order $d_1 \geq d_2 \geq \ldots \geq d_n$. We start with the following straightforward result.

Lemma. *The degree sequence of a sc-graph G on n vertices is symmetric about $(n-1)/2$, that is $d_i + d_{n+1-i} = n-1$. In particular,*

A. *if $n = 4k$, exactly half the vertices of G will have even degree, and exactly half will have odd degree;*

B. *if $n = 4k+1$, G will have at least one vertex of degree $(n-1)/2 = 2k$.*

Proof: A vertex v has degree d in G if and only if it has degree $n-1-d$ in $\overline{G}$. Thus G contains exactly r vertices of degree d iff $\overline{G}$ contains exactly r vertices of degree $n-1-d$, so the result follows. □

1.22. An isomorphism σ from a graph to its complement is called an *antimorphism*. Such a permutation maps $V(G)$ onto itself such that

A. $\forall v, w \in V(G),\ v \sim w$ in $G \Leftrightarrow \sigma(v) \sim \sigma(w)$ in $\overline{G} \Leftrightarrow \sigma(v) \not\sim \sigma(w)$ in G.

We note that σ is thus also an isomorphism from $\overline{G}$ to G.

If a set of vertices S is kept fixed by σ, that is $\sigma(S) = S$, then condition A will hold even when restricted to S. This means that S induces a self-complementary subgraph of G, with antimorphism $\sigma|_S$. In particular, we have

B. $\sigma(S) = S \Rightarrow |S| = 4s$ or $4s+1$ for some s.

In general, however, subgraphs or induced subgraphs of a sc-graph G are not themselves self-complementary (the most obvious example is the edge, K_2).

This means that there cannot be a forbidden subgraph or forbidden induced subgraph characterization of sc-graphs.

We will always express permutations as the product of disjoint cycles. It is easy to see that in each cycle of an antimorphism, the degrees of the vertices alternate between s and $n-1-s$ for some integer s. We can also find the parity of the number of vertices of each degree.

1.23. Lemma. *In a sc-graph G, the number of vertices of a given degree $d \neq (n-1)/2$ is even.*

Proof: Let G have exactly r vertices of degree $d \neq (n-1)/2$. Then G also has exactly r vertices of degree $n-1-d$, and any antimorphism must map these vertices onto each other. So the $2r$ vertices induce a self-complementary graph, and thus $2r \equiv 0 \pmod 4 \Rightarrow r \equiv 0 \pmod 2$. □

1.24. Lemma. *In a sc-graph G on $4k+1$ vertices, the number of vertices of degree $2k$ is $4s+1$ for some s.*

Proof: If G has r vertices of degree $2k$, they must map onto themselves under every antimorphism, so by 1.22.B we must have $r = 4s$ or $4s+1$ for some s. But if $r = 4s$ then, by 1.23 there are an even number of vertices of each degree, which is impossible. □

1.25. We now present Sachs' [341] and Ringel's [320] classical result on the structure of antimorphisms.

Theorem. *Every cycle of an antimorphism σ has length divisible by 4, except for a single fixed vertex whenever $n = 4k+1$.*

Proof: If $(v_1, v_2, \ldots, v_r)$ is a cycle of σ, then by 1.22.B we must have $r = 4s$ or $4s+1$.

Now, let $(v_1, v_2, \ldots, v_r)$ be a cycle of σ, with $t = 4s+1$ for some $s \geq 1$. Then $v_1 \sim v_2 \Leftrightarrow v_2 \not\sim v_3 \Leftrightarrow \cdots \Leftrightarrow v_{4s+1} \sim v_1 \Leftrightarrow v_1 \not\sim v_2$, which is a contradiction, so any odd cycles must be fixed vertices.

Moreover, an antimorphism cannot fix two vertices x and y, for then we would have $x \sim y \Leftrightarrow \sigma(x) \not\sim \sigma(y) \Leftrightarrow y \not\sim x$. So all cycles must have length a multiple of 4, except possibly for a single fixed vertex. Obviously, the total number of vertices is odd or even depending on the occurrence of this single odd cycle, so the result follows. $\square$

We note here that a given graph G will have many antimorphisms, possibly with different cycle lengths; if $n = 4k + 1$, these permutations may have different fixed vertices. Conversely, in the next chapter we will see that every permutation with appropriate cycle lengths is the antimorphism of some sc-graph, and we will present an algorithm for constructing all possible sc-graphs for each such permutation. In the meantime we look at sporadic constructions of sc-graphs which make use of antimorphisms.

New graphs from old

1.26. Given a graph G the generalized G-join of a family $F = (F_u | u \in V(G))$ of graphs is the graph H with vertex set $V(H) = \{(u, v) | u \in V(G), v \in V(F_u)\}$ and where

$$(u_1, v_1) \sim (u_2, v_2) \text{ if} \begin{cases} u_1 \sim u_2 \text{ in G, or} \\ u_1 = u_2 \text{ and } v_1 \sim v_2 \text{ in } F_{u_1}. \end{cases}$$

Basically, each vertex u of G is replaced by F_u, and each edge (u_1, u_2) of G is replaced by the bundle of all possible edges between F_{u_1} and F_{u_2}.

For some results on degrees, triangle numbers and diameters of G-joins and Cartesian products, see Nair [263, Sections 3.2, 3.3]. In particular, if G has diameter [radius] at least 2 then the diameter [radius] of any G-join will be the same as the diameter of G.

It is easy to see [Ruiz 1980] that if G is a self-complementary graph with antimorphism σ, and F a family of graphs such that $F_u \cong \overline{F}_{\sigma(u)}$, then the generalized G-join of F will also be self-complementary. In other words, for each cycle $(u_1, u_2, u_3, \ldots, u_{4r})$ of σ, we choose a graph L and replace the odd vertices by L and the even vertices by $\overline{L}$. We can choose a different L for each cycle, and we can choose L to be K_1 (to keep the vertices of the cycle unchanged) or K_0 (to remove the vertices of the cycle altogether). In the case of a fixed vertex (u), we replace u with a self-complementary graph.

This G-join will have an antimorphism closely mirroring the structure of σ. We replace each cycle $(u_1, u_2, u_3, \ldots, u_{4r})$ by a set of cycles mapping vertices of F_{u_i} onto corresponding vertices of $F_{u_{i+1}}$. Any one-cycle (u) is replaced by an antimorphism of F_u.

The G-join may have other antimorphisms. In particular, if all the vertices of a given cycle are replaced by copies of a self-complementary graph K, then we can replace the cycle $(u_1, u_2, u_3, \ldots, u_{4r})$ by an isomorphism mapping F_{u_1} onto $\overline{F}_{u_2}$, another (possibly different) mapping F_{u_2} onto $\overline{F}_{u_3}$ and so on.

For all we know, there might even be different but isomorphic G-joins. That is, the G-join of a family F might be isomorphic to the G'-join of F' for some other graph G' and some other family F'.

Notice that, given an odd order sc-graph G, with some fixed vertex v, we can produce larger sc-graphs of any feasible odd or even order by replacing v with a sc-graph of appropriate order. But if G has even order, the construction only allows us to produce even order sc-graphs. A quirk which we can only describe as, well, odd.

1.27. Since P_4 is the simplest non-trivial self-complementary graph, the P_4-join of $(G, \overline{H}, \overline{H}, G)$ is frequently used (see 1.65, 4.8 and 4.42). In 1.14 we used it to construct sc-graphs of diameter 3 without end-vertices. However, the construction of sc-graphs of diameter 2, and sc-graphs of diameter 3 with end-vertices was quite different — we took an arbitrary self-complementary graph on n vertices and added a P_4 to it in a specified manner. In this spirit we now give an alternative construction of sc-graphs of diameter 3 without end-vertices.

1.28. Let G be a sc-graph with $4k$ vertices, $k > 0$, and some antimorphism $\sigma = (u_1, u_2, u_3, u_4, \ldots, u_{4r})(u_{4r+1}, u_{4r+2}, \ldots, u_{4r+4s}) \cdots$. Define A, B, C, D to be the sets of vertices with subscripts congruent to 1, 2, 3, or 4 (mod 4), respectively, so that $\sigma(A) = B$, $\sigma(B) = C$, $\sigma(C) = D$ and $\sigma(D) = A$.

Now add a path $xvwy$, and join x, v, w and y to $A \cup D$, $A \cup B$, $D \cup C$ and $B \cup C$ respectively. The resulting graph G' (Figure 1.4) is a self-complementary graph of order $4k + 4$ with no end-vertices. Since x and y are not adjacent and do not have a common neighbour, G' cannot have diameter 2. (Alternatively, note that vw is a dominating edge, so that G' must have diameter 3).

Starting from $G_0 = P_4$, and repeatedly applying this procedure we obtain an infinite class of sc-graphs of diameter 3 without end-vertices for all feasible even $n \geq 8$.

We note that G' will depend on the numbering of the vertices in each

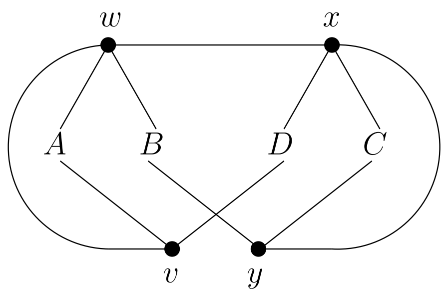


Figure 1.4: A sc-graph of order $4k+4$ and diameter 3 without end-vertices

cycle: for example, we could just as well number the first cycle

$$(u_{4r}, u_1, u_2, u_3, u_4, \ldots, u_{4r-1}),$$

and this would change A, B, C and D.

If we start with a self-complementary graph on $4k+1$ vertices, the antimorphism will have a single fixed vertex (z). We number all the other vertices so that z is adjacent precisely to the even numbered vertices (that is, to the vertices in $B \cup D$). We then add the path $xvwy$ as above, joining z to v and w (Figure 1.5).

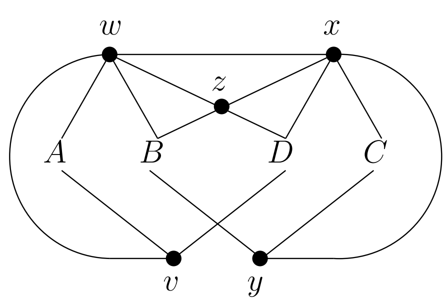


Figure 1.5: A sc-graph of order $4k+5$ and diameter 3 without end-vertices

We state this formally (and for the first time) below.

Lemma. *Every sc-graph on n vertices is an induced subgraph of a sc-graph of diameter 3 and order $n+4$ without end-vertices.*

Auto- and anti-morphisms

1.29. Let us denote the set of all automorphisms [antimorphisms] of a sc-graph G by $\mathcal{A}(G)$ [$\overline{\mathcal{A}}(G)$], or just $\mathcal{A}$ [$\overline{\mathcal{A}}$] where there is no ambiguity. We present here three fundamental results about antimorphisms, automorphisms, and the links between them. Part B and the first part of D of the following result were stated by Sachs [341], parts E, G and H by Gibbs [151], part C and the second part of D by Salvi-Zagaglia [343] and part H by Rao [310].

1.30. Theorem. *Let G be a self-complementary graph.*

A. *Let τ be a product of a finite number of automorphisms and antimorphisms of G. Then, depending on whether the number of antimorphisms is odd or even, τ will be an antimorphism or automorphism of G.*

B. *In particular, if $\sigma \in \overline{\mathcal{A}}$ then $\sigma^{2m} \in \mathcal{A}$ and $\sigma^{2m+1} \in \overline{\mathcal{A}}$ for any integer m.*

C. *The order of σ must be a multiple of 4, and thus G has at least two different antimorphisms.*

D. *The automorphism group of a sc-graph is non-trivial. In particular, G has a non-trivial automorphism with 0 or 1 fixed points (depending on whether $n = 4k$ or $4k+1$) and all other cycles of even length; and $\mathcal{A}(G)$ contains an involution.*

E. *G has an antimorphism whose cycle lengths are all powers of 2.*

F. *Let $\alpha \in \mathcal{A}$ be fixed. Then $\alpha\mathcal{A} = \mathcal{A}$ and $\alpha\overline{\mathcal{A}} = \overline{\mathcal{A}}$.*

G. *Let $\sigma \in \overline{\mathcal{A}}$ be fixed. Then $\sigma\overline{\mathcal{A}} = \mathcal{A}$ and $\sigma\mathcal{A} = \overline{\mathcal{A}}$, and both mappings are bijections. Thus, G has as many automorphisms as antimorphisms.*

H. *$\mathcal{A} \cup \overline{\mathcal{A}}$ forms a group under composition of functions, in which $\mathcal{A}$ is a normal subgroup of index 2.* □

Proof: Let $A'(G) = (a'_{ij})$ be the non-standard adjacency matrix of G, defined by

$$a'_{ij} = \begin{cases} 0, & \text{if } i = j, \\ 1, & \text{if } i \text{ is not adjacent to } j, \\ -1, & \text{if } i \text{ is adjacent to } j. \end{cases}$$

(This matrix was used by Gibbs [150, 151] in his study of sc-graphs, and was first introduced by Seidel). With each automorphism α we can associate a permutation matrix P_α so that $P_\alpha^T A'(G) P_\alpha = A'(G)$. With each antimorphism σ we can associate a permutation matrix P_σ so that $P_\sigma^T A'(G) P_\sigma = A'(\overline{G}) = -A'(G)$. Then P_τ is the product of all the associated permutation matrices, and we have $P_\tau^T A'(G) P_\tau = -A'(G)$ if the total number of antimorphisms is odd, and $P_\tau^T A'(G) P_\tau = A'(G)$ if the number is even.

We remark that B holds even for negative integers (because for any antimorphism σ, σ^{-1} is also an antimorphism). The first part of C follows directly from the cycle lengths of σ, and implies the second part of C and the first part of D. For $\sigma^2 \neq I$ is an automorphism, while $\sigma^3 \neq \sigma$ is another antimorphism.

For the second part of D, let r be the least integer such that $\sigma^r = I$. We can write $r = 2^s(2t+1)$ for some s, t. Then by C we must have $s \geq 2$. So if we define $\sigma' := \sigma^{2^{s-1}(2t+1)}$ then $\sigma' \neq I$, $\sigma'^2 = I$, and σ' is an automorphism since $s - 1 \geq 1$. So G has an automorphism whose cycles are all of length 1 or 2.

To prove E we define $\sigma'' = \sigma^{2t+1}$. Then σ'' is an antimorphism of order 2^s, and thus all the cycles of σ'' must have length a power of 2. Of course, none of the cycles will have length exactly 2, and there will be exactly one cycle of length 1 if and only if $n = 4k + 1$.

The fact that $\alpha\mathcal{A} \subseteq \mathcal{A}$, $\alpha\overline{\mathcal{A}} \subseteq \overline{\mathcal{A}}$, $\sigma\overline{\mathcal{A}} \subseteq \mathcal{A}$ and $\sigma\mathcal{A} \subseteq \overline{\mathcal{A}}$, follows from part A. To prove that, say, $\sigma\overline{\mathcal{A}} = \mathcal{A}$ we note that for any automorphism α_0, $\sigma^{-1}\alpha_0 \in \overline{\mathcal{A}}$ and $\sigma\sigma^{-1}\alpha_0 = \alpha_0$, so that the mapping $\phi \to \sigma\phi$, $\phi \in \overline{\mathcal{A}}$ is onto. Moreover, since σ always has an inverse, we have

$$\sigma\phi_1 = \sigma\phi_2 \Leftrightarrow \sigma^{-1}\sigma\phi_1 = \sigma^{-1}\sigma\phi_2 \Leftrightarrow \phi_1 = \phi_2$$

for all $\phi_1, \phi_2 \in \overline{\mathcal{A}}$, and thus the mapping is bijective.

The remaining parts of F and G are similarly proved, and then H follows as a consequence. □

1.31. We note that, unless all of G's antimorphisms have order 2^s (and maybe even then) it must have not only two different antimorphisms, but even two with different cycle structure. There are sc-graphs whose antimorphisms all have the same cycle structure — Hartsfield [198] gave an example of a sc-graph of order 13 whose antimorphisms all have cycles of lengths 1, 4, 8. Her main intention, though, was to display a graph (and a regular one at that) whose non-trivial cycles have unequal lengths; in 3.45 we extend her example to an infinite family of such graphs.

1.32. Rao proved the following in [310]:

Theorem. *Let* Γ *be a group. Then*

A. Γ *is isomorphic to the automorphism group of some self-complementary digraph* D.

B. Γ *is isomorphic to the automorphism group of some sc-tournament* T *if and only if* $|\Gamma|$ *is odd.*

C. If Γ *is the automorphism group of a sc-graph, then it must have even order.*

Proof: We will not prove part A. Part C is easy since every antimorphism σ has order a multiple of 4, and precisely the even powers of σ are automorphisms.

The proof we give of part B is due to Cameron [59]. It is known that a group Γ is isomorphic to the automorphism group of some tournament T if and only if Γ has odd order. To see that Γ must also be the automorphism group of a sc-tournament, take a copy of T and a copy of its converse T'. Then add an arc joining each vertex in T to the corresponding vertex of T' and, for every other pair of vertices, an arc from T' to T. □

Rao conjectured that the converse of C is true, but was unable to prove or disprove it. Cameron gave the following characterisation:

Theorem. *A group* Γ *is the automorphism group of some self-complementary graph if and only if there is a group* Ω *such that:*

A. Ω *contains a normal subgroup* $\Gamma' \cong \Gamma$,

B. $[\Omega : \Gamma'] = 2$, *and*

C. there is no element $\alpha \in \Omega - \Gamma'$ *such that* $\alpha^2 = \text{ID}$. $\square$

The necessity is obvious — if $\Gamma = \mathcal{A}(G)$ for some sc-graph G, just take Ω to be $\mathcal{A}(G) \cup \overline{\mathcal{A}}(G)$. As for sufficiency, Cameron said that "the converse is proved by a simple construction" but did not give any details. He did, however, add "Such a goup necessarily has even order, and it cannot be a complete group. Thus, for example, symmetric groups are excluded." So Rao's conjecture is not true, but we are still left with the problem of giving a more useful characterisation (and proof!) of the automorphism groups of sc-graphs.

1.33. Kotzig [227] defined the following sets and posed the problem of characterising them.

$$\begin{aligned} F(G) &:= \{u \in V(G) | \text{ there exists } \sigma \in \overline{\mathcal{A}}(G) \text{ such that } \sigma(u) = u\}, \\ N(G) &:= \{uv \in E(G) | \text{ there exists } \sigma \in \overline{\mathcal{A}}(G) \text{ such that } \sigma(u) = v\}. \end{aligned}$$

Rao [306] not only solved this problem, but also showed how $V(G)$ is partitioned into equivalence classes (which we call orbits) under $\mathcal{A}(G)$. (For details on other problems due to Kotzig see 3.37).

We use $u \xrightarrow{f} v$ to mean that u and v are in the same orbit, and f is an automorphism such that $f(u) = v$.

1.34. Theorem. *Let G be a self-complementary graph. Then the orbits of G can be numbered $V_1, V_2, \ldots, V_{2s}$ if $n = 4k$, or $V_0, V_1, V_2, \ldots, V_{2s}$ if $n = 4k+1$ such that*[2]*:*

A. $|V_0| = 4t + 1$ *for some t, and $|V_i|$ is even for all $i \geq 1$.*

B. $\sigma(V_0) = V_0$ *and* $\sigma(V_i) = V_{2s+1-i}$ *for any antimorphism σ;*

C. $G[V_0]$ *is a regular sc-subgraph (of degree $(|V_0|-1)/2$, and $G[V_i, V_{2s+1-i}]$ is a regular bipartite self-complementary subgraph (of degree $|V_i|/2$) for all $i \geq 1$.*

D. $F(G) = V_0$.

E. $N(G) = E[V_0] \cup \bigcup_{i=1}^{s} E[V_i, V_{2s+1-i}]$.

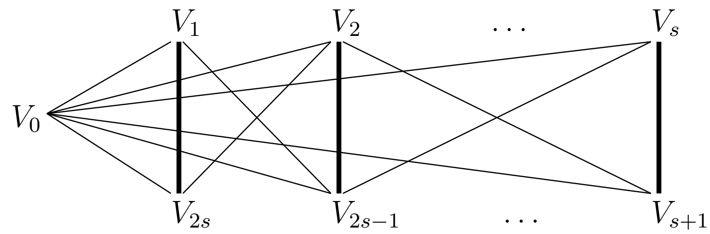


Figure 1.6: The orbits of a sc-graph; the bold lines denote the edges of $N(G)$.

Proof: A. Let σ be an arbitrary antimorphism of G, which we can sketch as $(u_0)(u_1 \dots u_{4r}) \cdots (u_{4s+1} \dots u_{4k})$. In any given cycle, all the odd-numbered vertices are in a single orbit of G, while the even-numbered vertices are also in a single orbit (possibly the same as that of the odd-numbered vertices). Thus any orbit V_i will contain an even number of vertices from each cycle, except for a single orbit, which we denote V_0, which also contains the vertex u_0. Thus all orbits have even size except for V_0, which has odd size. (We prove later on that $|V_0| \equiv 1 \pmod 4$).

D. The fixed vertex of any antimorphism is contained in the unique orbit of odd size, that is $F(G) \subseteq V_0$. Now, if $n = 4k+1$, there must be at least one vertex $u \in F(G) \subseteq V_0$, say $\sigma(u) = u$. Let v be any other vertex in V_0, and let $u \xrightarrow{f} v$. Then $f\sigma f^{-1}$ is an antimorphism which fixes v. Thus $F(G) = V_0$; we denote it sometimes by F for simplicity.

B. If $u \xrightarrow{f} v$, then $\sigma(u) \xrightarrow{f'} \sigma(v)$, where $f' = \sigma f \sigma^{-1}$. Conversely, if $\sigma(x) \xrightarrow{g} \sigma(y)$, then $x \xrightarrow{g'} y$, where $g' = \sigma^{-1} g \sigma$. Thus every antimorphism induces a bijection on the orbits of G. We now show that each antimorphism must induce the same bijection. For if σ_1, σ_2 are antimorphisms, then $\sigma_1(u) \xrightarrow{h} \sigma_2(u)$ where $h = \sigma_2\sigma_1^{-1}$; that is, both antimorphisms must map u to the same orbit.

Now V_0 is the unique odd-sized orbit, so we must have $\sigma(V_0) = V_0$. This proves the first part of B, and also shows us that V_0 must induce a sc-subgraph, and we thus have $|V_0| = 4t+1$ for some t, which completes the proof of A. Further, V_0 forms a single orbit under $\mathcal{A}(V_0)$, so that $G[V_0]$ must be vertex-transitive, and thus regular; this proves the first part of C.

[2]The results and proofs are stated for $n = 4k+1$. The case $n = 4k$ is analogous and simpler, as any references to V_0 or fixed vertices should just be ignored. See Figure 1.6 for an illustration.

Since σ^2 is an automorphism, we must have $\sigma^2(V_i) = V_i$ for every orbit, that is $\sigma(V_i) = V_j \Leftrightarrow \sigma(V_j) = V_i$. We now show that we cannot have an even-sized orbit V_i with $\sigma(V_i) = V_i$. For then V_i would induce a sc-subgraph on $4r$ vertices, for some r, and the induced bipartite subgraph $G[V_0, V_i]$ would have $(4t+1)2r$ edges. Since every vertex of V_i has the same degree in $G[V_0, V_i]$, this is impossible. So the even-sized orbits must be paired up as stated in B (incidentally, this also shows that the number of even-sized orbits is in fact $2s$, for some s).

Since $|V_i| = |V_{2s+1-i}|$, we can see that $G[V_i, V_{2s+1-i}]$ must satisfy the conditions stated in C.

We turn finally to E. By virtue of B, we have that $N(G)$ is a subset of the right hand side, and we only have to prove equality. Let $uv \in E[V_0]$, let σ be an antimorphism which fixes u, and $u \xrightarrow{f} v$ for some f. Then $f\sigma$ is an antimorphism with $f\sigma(u) = v$, and thus $uv \in N(G)$. Now let uv be an edge with $u \in V_i$, $v \in V_{2s+1-i}$ for some $i \geq 1$; choose an arbitrary antimorphism σ, let $\sigma(u) = w$ and $u \xrightarrow{g} v$ for some g. Then $g\sigma$ is an antimorphism with $g\sigma(u) = v$, so $uv \in N(G)$. □

1.35. Corollary. *The following are equivalent for a sc-graph G:*

- *G is vertex-transitive*
- *$F(G) = V(G)$*
- *$N(G) = E(G)$.* □

The structure of sc-graphs

1.36. The characterisation of $F(G)$ is very important as it gives us a natural association between sc-graphs on $4k+1$ vertices and sc-graphs on $4k$ vertices, though unfortunately not *vice versa*. In fact, the result that $F(G)$ is the unique odd orbit of G was essentially first proved by Robinson [321] (and also by Molina [255], much later but apparently independently of the other two authors). Robinson further proved [324] that for all sc-digraphs $F(G)$ is the unique orbit fixed by any and every antimorphism, and this allows us to extend Rao's theorem to self-complementary digraphs (see 5.7 for the details).

Lemma. *If G is a sc-graph on $4k+1$ vertices, and $v \in F(G)$, then $G - v$ is self-complementary. If w is any other vertex in $F(G)$, then $G - v \cong G - w$.*

Proof: Since $v \in F(G)$, there is at least one antimorphism which fixes v, and which thus fixes $G - v$. Then $G - v$ must be self-complementary. Since $F(G)$ forms an orbit of G, there is an automorphism α such that $\alpha(v) = w$ and thus $\alpha(G - v) = G - w$. $\square$

We will call $G - v$ a maximal fixed subgraph of G, or *the* maximal fixed subgraph if we are only interested in its isomorphism type. The lemma tells us that every odd order sc-graph has a unique maximal fixed subgraph, up to isomorphism. However, we cannot obtain G from its maximal fixed subgraph since we do not know which vertices were neighbours of the removed vertex.

1.37. Definition. *Let G be a self-complementary graph of order $4k$, with some antimorphism σ. Let A, B be disjoint sets of $2k$ vertices each, such that $\sigma(A) = B$ and $\sigma(B) = A$. We say that A and B are exchanged by σ, and that A and B are exchangeable sets. The subgraphs induced by A and B are called exchangeable subgraphs. The graph $G(A, B)$ is constructed by adding a vertex v_0 to G, and joining it to all the vertices of A.*

1.38. Lemma. *G is a maximal fixed subgraph of H if and only if $H = G(A, B)$, for some exchangeable sets A, B.*

Proof: If A and B are exchanged by σ, then $(v_0)\sigma$ will be a antimorphism of $G(A, B)$, with G as a maximal fixed subgraph.

Conversely, let G be a maximal fixed subgraph of some graph H, and let $\phi = (w)\sigma$ be the antimorphism fixing G. Let A and B be the sets of neighbours and non-neighbours of w. Then obviously σ is an antimorphism of G which exchanges A and B. $\square$

1.39. If we take Lo [Hi] to be the set of vertices of G of degree less than $2k$ [at least $2k$], then $G(Lo, Hi)$ and $G(Hi, Lo)$ are non-isomorphic sc-graphs both of which have G as maximal fixed subgraph. So an even order sc-graph is never the maximal fixed subgraph of a unique graph.

We note that the graphs $G(Hi, Lo)$ have exactly one vertex of degree $2k$, and thus $|F(G(Hi, Lo))| = 1$. However, even regular graphs can have $|F(G)| = 1$ [338] — for any k, the C_5 join of $(\overline{K}_k, K_k, K_1, K_k, \overline{K}_k)$ is regular but the central K_1 is unique as it is the only vertex which is in just two cliques.

Problems.

A. *When can a self-complementary graph G contain a self-complementary subgraph H which is not fixed by any antimorphism of G?*

B. *Can this occur when $|V(G)| = 4k+1$ and $|H| = 4k$?*

C. *If so, will H be isomorphic to the maximal fixed subgraph of G?*

We refer to 1.16 for a related discussion, and note that the Decomposition Theorem stated there will probably help in answering A.

A pair of vertices v, w are said to be pseudo-similar if there is no automorphism mapping v to w, but $G - v \cong G - w$. So the last problem can be rephrased as follows:

C. *When can a sc-graph G contain a pair of pseudo-similar vertices v, w with $v \in F(G), w \notin F(G)$?*

We now look at two notable cases where there *is* a bijection between classes of odd order and even order sc-graphs.

1.40. Theorem[Robinson 1969]**.** *The sc-graphs on $4k$ vertices are not Eulerian, but they are in one-one correspondence with the Eulerian sc-graphs on $4k+1$ vertices.*

Proof: We note that, since sc-graphs are connected, the Eulerian sc-graphs are precisely those where every vertex has even degree. Now if G is a sc-graph of order $4k$, exactly half its vertices will have odd degree and exactly half will have even degree, so that G is not Eulerian.

However, if we denote the sets of vertices of odd and even degrees by A and B, respectively, then A and B are exchangeable and $G(A, B)$ is Eulerian. These are obviously the only sets which will give us an Eulerian graph, so each even order sc-graph G is the maximal fixed subgraph of exactly one Eulerian sc-graph. Since every (Eulerian) sc-graph on $4k+1$ vertices has a unique maximal fixed subgraph, the correspondence is established. $\square$

1.41. Obviously, whenever we have a unique way of identifying a pair of exchangeable sets, we can set up a similar bijection. For example, self-complementary graphs with loops allowed can exist only on $4k$ vertices, and must have exactly $2k$ loops; the sets of vertices with and without loops are exchangeable, and so there is a bijection between sc-graphs on $4k$ vertices with loops and sc-graphs on $4k+1$ vertices.

An *almost regular graph* is a graph whose vertices have one of two degrees, s and $s-1$, for some s; we insist that at least one vertex have degree s, and at least one vertex degree $s-1$, otherwise we get a regular graph. The following is partly due to Sachs [341].

Lemma. *A regular self-complementary graph G must have $4k+1$ vertices and degree $2k$ for some k, and diameter 2. An almost regular sc-graph H must have $4k$ vertices, of which half have degree $2k$ and half $2k-1$, for some k. Moreover, the regular and almost regular sc-graphs are in one-one correspondence.*

Proof: By Lemma 1.21 the degree of a regular sc-graph G on n vertices must be $r = (n-1)/2$. For r to be an integer, $n-1$ must be even, and therefore we must have $n = 4k+1$ and $r = 2k$ for some k.

To show that G has no vertices at distance 3 or more, let v and w be non-adjacent vertices of G. Of the remaining $4k-1$ vertices, $2k$ are neighbours of v, and $2k$ are neighbours of w. By the Pigeonhole Principle v and w must then have a common neighbour.

Now let H be an almost regular sc-graph on n vertices. By Lemma 1.21 $s+(s-1) = 2s-1 = n-1$. Then $n = 2s$, so we must have $s = 2k$ for some k; since n is even, exactly half the vertices must have degree s and half $s-1$.

The last part is just a special case of 1.40, since regular sc-graphs are also Eulerian: the maximal fixed subgraph of a regular sc-graph is almost regular, and as in 1.40 each almost regular sc-graph H is the maximal fixed subgraph of exactly one regular (Eulerian) sc-graph. $\square$

1.42. We can define a concept similar to exchangeable sets for any even-order graph. Let H be a graph on $2k$ vertices with some automorphism α. If H_1 and H_2 are disjoint sets of k vertices each, such that $\alpha(H_1) = H_2$ and $\alpha(H_2) = H_1$, then we say that H_1 and H_2 are *interchangeable sets*, and that they are interchanged by α; the subgraphs induced by H_1 and H_2 are called *interchangeable subgraphs*.

It is easy to see that a pair of exchangeable subgraphs must be complements of each other, and must each contain a pair of interchangeable sets.

P.S. Nair [266] showed that the converse is also true. Let H be a graph with interchangeable sets $H_1 = \{v_1, v_2, \ldots, v_k\}$ and $H_2 = \{\alpha(v_1), \alpha(v_2), \ldots, \alpha(v_k)\}$; we will construct a sc-graph containing H and $\overline{H}$ as exchangeable subgraphs. Take a copy of $\overline{H}$, with vertices $\{\overline{v}_1, \overline{v}_2, \ldots, \overline{v}_k\}$ and $\{\alpha(\overline{v}_1), \alpha(\overline{v}_2), \ldots, \alpha(\overline{v}_k)\}$.

Now define the first cycle of a permutation σ to be

$$(v_1\overline{v}_1\alpha(v_1)\alpha(\overline{v}_1)\alpha^2(v_1)\alpha^2(\overline{v}_1)\alpha^3(v_1)\alpha^3(\overline{v}_1)\cdots).$$

Let j be the least integer for which $v_1 = \alpha^j(v_1)$. Then $j = 2s$ for some s, and the cycle has length $4s$. If there is a vertex v_2 not included in this cycle, we can define another cycle as above, and so on until σ includes all vertices of H and $\overline{H}$. All cycles will have length a multiple of 4 and so we can construct a bipartite self-complementary graph B with bipartite antimorphism σ. Then $H \cup B \cup \overline{H}$ (fig is a self-complementary graph with antimorphism σ.

Nair claims, without proof, that in this way we can construct all self-complementary graphs on $4k$ vertices containing H and $\overline{H}$ as vertex-disjoint subgraphs. In particular, he claims that if H' is a graph on $2k$ vertices which does *not* have an automorphism α interchanging some subgraphs H_1' and H_2' on k vertices, then there is no sc-graph G on $4k$ vertices containing vertex-disjoint copies of H' and $\overline{H'}$. This is not obviously true, and so we list it as an open question:

Problem. *If a sc-graph G on $4k$ vertices contains H and $\overline{H}$ as vertex-disjoint induced subgraphs, must these subgraphs be exchanged by some antimorphism? If not, must H at least contain a pair of interchangeable sets? What if $|V(H)| = 2k$?*

1.43. Molina [255] studied the structure of odd order sc-graphs more closely, with a view to generating them efficiently. Let v be in $F(G)$, σ be an antimorphism fixing v, X be the set of neighbours of v, and $Y = V(G) - X - \{v\}$. Then $\sigma(X) = (Y)$ and $\sigma(Y) = X$, so that $G[X] = \overline{G[Y]}$. Moreover, σ maps the induced bipartite graph $G[X, Y]$ with partition $P = (X, Y)$ onto its bipartite complement $K_{|X|,|Y|} - G[X, Y]$. (We have to specify the partition explicitly since disconnected bipartite graphs do not have a unique bipartite complement.)

We say that $G[X, Y]$ is a bipartite self-complementary graph with respect to $K_{|X|,|Y|}$. Since $\sigma|_{G[X,Y]}$ switches the sets of the partition, we call it a mixed bipartite antimorphism. Bipartite self-complementary graphs are treated in more detail in Chapter 5; in particular, it is proved that a mixed bipartite antimorphism must have cycles whose length is a multiple of 4.

So if A, B are graphs such that $A \cong G[X]$, and $B \cong G[X, Y]$, we can sketch G as in Figure 1.7.

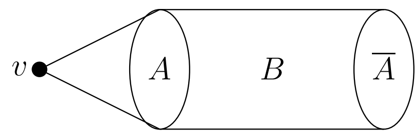


Figure 1.7: The structure of sc-graphs

We say that G is of type (A, B); while this tells us explicitly which vertices of G are neighbours of the removed vertex v, it does not tell us what the maximal fixed subgraph is. Given a graph A on $2k$ vertices, and a bipartite self-complementary graph B on $4k$ vertices, there may be many ways of superimposing copies of A and $\overline{A}$ on to the parts of B to get a sc-graph on $4k$ vertices.

So we have gained something and lost something. Just as we can have two non-isomorphic odd order sc-graphs with the same maximal fixed subgraphs, so we can have two non-isomorphic sc-graphs with the same type. However, if we specify both the type and the maximal fixed subgraph, then we know what G is.

1.44. Molina used his analysis to generate odd order sc-graphs systematically, using the following construction:

1. Take a (labelled) bipartite self-complementary graph B with parts X and Y of size $2k$ each.

2. Take a bipartite antimorphism ϕ mapping X to Y and Y to X.

3. Take a (labelled) graph A with vertex-set X such that $\phi^2(A) = A$; define $C := \overline{\phi(A)} = \phi(\overline{A})$.

4. Construct the graph $G_{A,B} = v + A \cup B \cup C$, where $v + A$ means that v is joined to all the vertices of A. Then $G_{A,B}$ is a self-complementary graph on $4k + 1$ vertices with $\sigma := (v)\phi$ as an antimorphism.

This construction has to be carried out for all labelled graphs A, and all bipartite self-complementary graphs B which have a mixed antimorphism. Isomorphism checks only need to be carried out between $G_{A,B}$ and $G_{A',B'}$ when $A \cong A'$ and $B \cong B'$, and one only needs to check for isomorphisms which fix v, X and Y.

The procedure can stop as soon as the number of non-isomorphic graphs generated equals the number of self-complementary graphs on $4k+1$ (see 7.8 for a counting formula). If all possible graphs A, B have been used but there are still sc-graphs missing, it will be necessary to repeat the procedure with different permutations in step 2. However, Molina generated the 36 sc-graphs on 9 vertices using just one permutation for each choice of B. He recommended using permutations with as many 4-cycles as possible.

Parthasarathy and Sridharan [285] gave a formula for the number of sc-graphs with a given degree sequence. This may be incorporated into the procedure, as once all the sc-graphs with a given degree sequence have been generated, one can exclude any further graphs with the same degree sequence, without the need for an isomorphism check.

1.45. We study antimorphisms and generation methods further in 4.12–4.18, where we show how to construct systematically the self-complementary graphs corresponding to a given permutation. In 2.11–2.12 we also outline a method of proof which relies on basic properties of antimorphisms.

Planarity and thickness

1.46. The first few sc-graphs (Figure 1.1 are all planar, but they are the exception rather than the rule. In fact quite a few of the results on sc-graphs hold only for n sufficiently large, and the reader is cautioned from using the first few sc-graphs as a representative sample. Sc-graphs in general have too many edges to be planar, and the same is true for embeddings on other surfaces.

Lemma. *For any constant c, there are only a finite number of sc-graphs with orientable genus $g(G) \le c$ or thickness $t(G) \le c$. In particular, every sc-graph on at least 9 vertices is non-planar.*

Proof: By the Ringel-Youngs and Euler formulas (c.f. [387, 13G, 14C, 14D] the genus of a graph G with $n \geq 4$ vertices is bounded by $\left\lceil \frac{(n-3)(n-4)}{12} \right\rceil \geq g(G) \geq \left\lceil \frac{|E(G)|-3n+6}{6} \right\rceil$, while the thickness is bounded by $t(G) \geq \left\lceil \frac{|E(G)|}{3(n-2)} \right\rceil$.

For sc-graphs this gives us

$$\left\lceil \frac{(n-3)(n-4}{12} \right\rceil \geq g(G) \geq \left\lceil \frac{n^2-13n+24}{24} \right\rceil$$

and

$$t(G) \geq \left\lceil \frac{n(n-1)}{12(n-2)} \right\rceil .$$

Both inequalities tell us that G is definitely non-planar when $n^2 - 13n + 24 \geq 0$, that is when $n \geq 11$. This was improved to $n \geq 9$ in [32]. □

1.47. Let $g_M(G)$ be the maximum genus of the orientable surfaces on which G has a 2-cell embedding. It is known that $g_M(G) \leq \frac{1}{2}(n - |E(G)| + 1)$; if the upper bound is reached, G is said to be upper embeddable. It was proved by Nebeský [271] and Scotti [349] that either G or $\overline{G}$ must be upper embeddable, so we have the following:

Theorem. *All self-complementary graphs are upper embeddable.* □

We briefly discuss sc-graphs with topologically self-dual maps in 3.25.

Eigenvalues

1.48. We saw in 1.30 how the non-standard adjacency matrix can be useful in reasoning about self-complementary graphs. Gibbs studied the eigenvalues of this adjacency matrix in [150, 151], providing the following results and a conjecture.

Theorem. *If G is a graph on $2k$ vertices (not necessarily self-complementary), and x an eigenvalue of its non-standard adjacency matrix $A'(G)$, then $\frac{x^2-1}{2}$ is an algebraic integer; in particular $A'(G)$ is non-singular, and if x^2 is* *an integer then x is odd. If G is a graph on $2k+1$ vertices, then $A'(G)$ has rank at least $2k$.* □

1.49. Theorem. *Let G be a self-complementary graph with $n = 4k$ or $4k+1$ vertices and non-standard adjacency matrix $A'(G)$. Then*

A. *$A'(G)$ has exactly $4k$ non-zero eigenvalues, and one zero eigenvalue if $n = 4k+1$.*

B. *The non-zero eigenvalues occur in opposable pairs, $\pm a_1, \pm a_2, \ldots, \pm a_{2k}$, and $\sum_{i=1}^{2k} a_i^2 = \binom{n}{2}$.* □

1.50. Theorem. *If a sc-graph G on $4k$ vertices has an antimorphism with cycles of equal length, and non-standard adjacency matrix $A'(G)$, then*

A. *$A'(G)$ is similar to a symmetric $2k \times 2k$ matrix A_1, whose diagonal entries are s or $-s$, and where all other entries are $\pm e \pm s$, where e denotes the 2×2 identity matrix, and*

$$s = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

B. *If from A_1 we form the $2k \times 2k$ matrix B_1 by setting $e = s = 1$, and B_2 by setting $e = -s = 1$, then the eigenvalues of $A'(G)$ are precisely those of B_1 together with those of B_2.*

C. *If x is an eigenvalue of $A'(G)$, then $(x^2-1)/4$ is an algebraic integer. In particular, if x^2 is an integer, then $x^2 \equiv 1 \pmod 4$.* □

1.51. Conjecture. *The results of 1.50.C hold for any sc-graph on $4k$ vertices.*

1.52. What can we say about the eigenvalues of the usual $0-1$ adjacency matrix of self-complementary graphs? There is a certain degree of symmetry in the spectrum ensured by the following result of Cvetković [97, 99, Sec. 2.2]. The final identity is also mentioned by Godsil and McKay [156], though in [154, Ex. 4.10] it is reported with a misplaced bracket.

Theorem. *If λ is an eigenvalue of G with multiplicity $m > 1$, then $-\lambda - 1$ will be an eigenvalue of $\overline{G}$ with multiplicity either $m-1$, m, or $m+1$. If $A = A(G)$, J is the matrix of all 1's, and j is a column of J, we also have*

$$\frac{f_{\overline{G}}(\lambda - 1)}{(-1)^n f_G(-\lambda)} = |I - (\lambda I + A)^{-1} J| = 1 - j^T (\lambda I + A)^{-1} j.$$

This expression is also equal to

$$1 - \frac{1}{\lambda} W_G(\frac{-1}{\lambda})$$

where $W_G(\lambda)$ is the generating function for the number of walks in G. □

1.53. We can say more in the case of regular graphs. Sachs' seminal 1962 paper included a study of the characteristic polynomial of regular graphs, $f_G(\lambda) = |\lambda I - A(G)|$. It is known (c.f. [42, Prop. 3.1] that, for a connected regular graph G of degree r and order n, $f_G(\lambda)$ has r as a simple root. Sachs proved, moreover, that

$$(\lambda + r + 1) f_{\overline{G}}(\lambda) = (-1)^n (\lambda - n + k + 1) f_G(-\lambda - 1).$$

For a regular sc-graph on $4k+1$ vertices, this gives

$$(\lambda + 2k + 1) f_G(\lambda) = (2k - \lambda) f_G(-\lambda - 1).$$

Writing the roots as $\lambda_1, \ldots, \lambda_{4k}, 2k$ we get:

Theorem[Sachs 1962]**.** *If G is a rsc-graph on $4k+1$ vertices, then its characteristic polynomial can be written as:*

$$f_G(\lambda) = (\lambda - 2k) \prod_{i=1}^{2k} (\lambda - \lambda_i)(\lambda + \lambda_i + 1). \square$$

1.54. Sachs also proved the following theorem which is not restricted to sc-graphs, but gives an idea of the wealth of results this single paper contains:

Theorem. *If G is a connected circulant graph on n vertices of degree r, then for every divisor d of n there are natural numbers u_d, v_d with $u_d v_d = \phi(d)$, and an irreducible monic polynomial p_d of degree u_d such that*

$$f_G(\lambda) = \prod_{d|n} [p_d(\lambda)]^{v_d}.$$

Moreover, $p_1(\lambda) = \lambda - r$ *and* $v_1 = 1$*. If* n *is even,* $v_2 = 1$*; while if* n *is odd,* $v_d = 2c_d > 0$ *for some natural number* c_d*.* □

1.55. It follows from Beineke [33] that there are only six self-complementary line graphs — K_1, P_4, C_5, the A-graph, an almost regular sc-graph H_8 on 8 vertices, and a vertex-transitive sc-graph H_9 on 9 vertices (see Figure 1.8. Incidentally, C_5 and H_9 are the Paley graphs on 5 and 9 vertices, respectively, while P_4 and H_8 are their maximal fixed subgraphs.

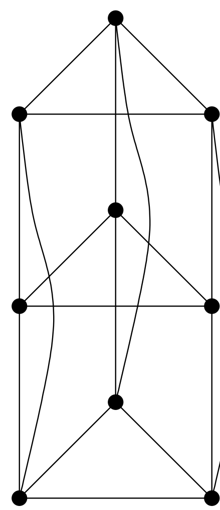


Figure 1.8: The largest self-complementary line graph

Radosavljević and Simić have shown that these six graphs are also the self-complementary generalised line graphs [295], that the first four are the only sc-graphs which can be oriented to become a line digraph [296], and the sc-graphs whose eigenvalues are all at least -2 have at most 13 vertices.

Let $\mathcal{A}(G)_{v_1,\ldots,v_k}$ denote the group of automorphisms of G which fix the vertices v_1 through v_k individually. Then a graph G is said to be stable if its vertices can be ordered $v_1, \ldots, v_n$, so that, for $1 \leq k \leq n$, the automorphism group of $G - \{v_1, \ldots, v_k\}$ is $\mathcal{A}(G)_{v_1,\ldots,v_k}$. Grant [161] showed that there are only a finite number of graphs H such that $L(H)$ and $L(\overline{H})$ are both stable, and that $H = P_4$ is the only one that is self-complementary.

There are some restricted results about the underlying graphs of self-converse line digraphs in [376]. It is proved in [5] that the graphs whose complement and line graph are isomorphic are just the pentagon, and the triangle with three independent endvertices; while those digraphs with isomorphic converse and line digraph are determined in [204]. In particular, the self-converse digraphs which are isomorphic to their line digraph are just the circuits [260]. For further results and remarks about line graphs see 2.22 and 5.2.

1.56. Because of the importance of the isomorphism and recognition problem for self-complementary graphs, it would be interesting to know when the spectrum of a sc-graph is unique, or whether the class of sc-graphs is collectively characterised by their spectra. The problem here is the existence of pairs of co-spectral graphs. Of course, a sc-graph is co-spectral to its complement, and maybe this limits the range of other graphs to which it can be co-spectral.

Problems. *Can a self-complementary graph G be co-spectral to a graph H which is not self-complementary? If such a graph exists, must it also be co-spectral to $\overline{H}$? When can a self-complementary graph G be co-spectral to another self-complementary graph?*

The following result of Johnson and Newman [214] may be useful in investigating these problems — if G and H are co-spectral, then $\overline{G}$ and $\overline{H}$ are co-spectral if and only if there is an orthogonal matrix L, with row and column sums all equal to 1, such that

$$L^T A(G)L = A(H) \text{ and } L^T A(\overline{G})L = A(\overline{H}).$$

(In fact the two equations are equivalent because $LJ = JL = J$, where J is the all ones matrix).

Chromaticity

1.57. In one of the shortest and best known papers in graph theory, Nordhaus and Gaddum [274] proved the following bounds on the chromatic number of a graph, and showed that they are attained by infinitely many graphs:

$$\sqrt{n} \le \sqrt{\chi(G)\chi(\overline{G})} \le \frac{\chi(G) + \chi(\overline{G})}{2} \le \frac{n+1}{2}.$$

Noting that a graph G is r-partite if and only if $\chi(G) = r$, this gives us:

Theorem. *Let G be a self-complementary graph. Then*

$$\sqrt{n} \leq \chi(G) \leq \tfrac{n+1}{2}.$$

In particular, for any constant r, there are only a finite number of self-complementary graphs which are r-partite. □

1.58. Theorem[Chao and Whitehead 1979]**.**

A. *For every m there is a sc-graph with $\chi(G) = m$ attaining the lower bound of Theorem 1.57.*

B. *For every $m \geq 2$ there is a sc-graph of diameter 3 with $\chi(G) = m$ attaining the upper bound of Theorem 1.57.*

C. *For every $m \geq 3$ there is a sc-graph of diameter 2 with $\chi(G) = m$.*

Proof: We will only prove B and C. We construct an infinite class of sc-graphs U_n as follows. When $n = 4k$, $V(U_n) = \{1, 2, \ldots, 4k\}$. The vertices $1, 2, \ldots, 2k$ induce a complete graph; while $2k + 1$ is adjacent to $\{1, 2, \ldots k\}$, $2k + 2$ is adjacent to $\{2, 3, \ldots, k + 1\}$, and so on. This is illustrated in Figure 1.9. When $n = 4k+1$, we add a vertex $4k+1$ and join it to $1, 2, \ldots, 2k$.

The graph U_n has diameter 3 because the vertices $3k$ and $4k$ are not adjacent and have no common neighbour. It can also be seen that

$$\chi(U_n) = \begin{cases} 2k, & \text{if } n = 4k \\ 2k+1, & \text{if } n = 4k+1. \end{cases}$$

We now construct W_n, $n \geq 8$. To each U_{n-4}, add a P_4 as in 1.13 (see Figure 1.2), joining the end-vertices to each vertex of U_{n-4}. W_n is a self-complementary graph of diameter 2 and for which

$$\chi(W_n) = \chi(U_{n-4}) + 1 = \begin{cases} 2k-1, & \text{if } n = 4k \\ 2k, & \text{if } n = 4k+1. \end{cases}$$

Finally, for $n = 5$ we define W_5 to be the pentagon. □

1.59. A matching is a subgraph consisting of disjoint edges. Let $a_G(r)$ be the number of matchings of G with r edges; then we define the matchings

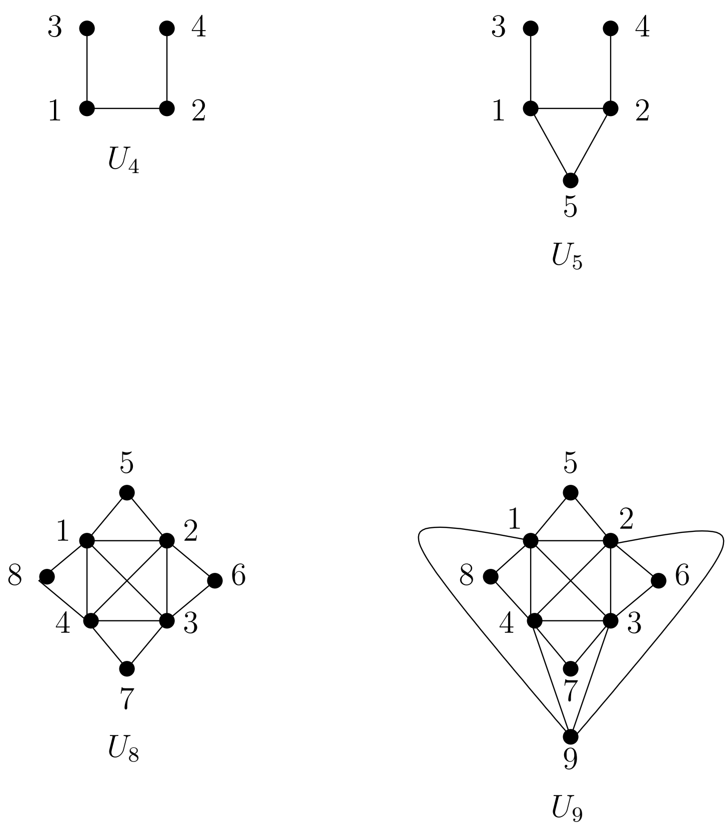


Figure 1.9: The graphs U_n, $n = 4, 5, 8, 9$

polynomial of G to be

$$M(G; w) = \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} a_G(r)\ w^{n-r}.$$

Let $P(G; \lambda)$ be the chromatic polynomial of G, which represents the number of ways of colouring G with λ colours such that no two adjacent vertices receive the same colour. If we define $b_G(r)$ to be the number of partitions of $V(G)$ into $n - r$ non-empty sets, each of which induces a null graph, and $(\lambda)_r := \lambda(\lambda - 1)(\lambda - 2) \cdots (\lambda - r + 1)$, then we can express $P(G; \lambda)$ as

$$P(G; \lambda) = \sum_{r=0}^{n} b_G(r)\ (\lambda)_{n-r}.$$

Obviously, if G is self-complementary then $P(G) = P(\overline{G})$; we say that G and $\overline{G}$ are chromatically equivalent. But Koh and Teo [225], Liu, Zhou and Tan [238] and Xu and Liu [395] have shown that for all $n \equiv 0, 1 \pmod 4), n \geq 8$, there exist graphs which are not self-complementary, but which are nonetheless chromatically equivalent to their complements. This answered a question of Akiyama and Harary [13].

Theorem[Farrell and Whitehead 1992]**.** *Let G be a self-complementary graph, and $M(G; w)$, $P(G; \lambda)$, be as above. Then $b_G(r) \geq a_G(r)$ for all r; and there is at least one r for which $b_G(r) > a_G(r)$ if and only if G contains a triangle, that is $G \notin \{K_1, P_4, C_5\}$. Moreover [Godsil 1981, Zaslavsky 1981, Wahid 1983],*

$$M(G; w) = \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^r a_G(r)\ w^r M(K_{n-2r}; w). \square$$

1.60. Theorem[Gutman 1980, Farrell and Whitehead 1992]**.** *Let G be a bipartite self-complementary graph with respect to $K_{m,n}$, and $M(G; w)$, $P(G; \lambda)$, be as above. Then*

$$M(G; w) = \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^r a_G(r)\ w^r M(K_{m-k,n-k}; w). \square$$

1.61. Some authors (c.f. Godsil [154]) define the matchings polynomial as

$$\mu(G; x) := \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^r a_G(r) x^{n-2r}.$$

It follows from Godsil (ibid., p. 97, 107) and Clapham [83] and Camion [61] that when G is self-complementary, the n roots of $\mu(G, x)$ are all distinct and real. The number of perfect matchings is denoted by $\mathrm{pm}(G)$, where a perfect matching is a set of $\frac{n}{2}$ edges which contain each vertex exactly once. Obviously $\mathrm{pm}(G) = 0$ when G has odd order. We also have the rook polynomial

$$\rho(G; x) := \sum_{r=0}^{n} (-1)^r a_G(r) x^{n-r}.$$

With these definitions we have the following:

Theorem[Godsil 1993, p. 6, 7]**.** *If G is a self-complementary graph then*

$$\mu(G,x) = \sum_{r=0}^{n} a_G(\tfrac{n-r}{2})\mu(K_r,x) = \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} a_G(r)\mu(K_{n-2r},x), \text{ and}$$

$$pm(G) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2/2}\mu(G,x)\ dx.\Box$$

1.62. Theorem[Joni and Rota 1980, Godsil 1981]**.** *If G is a bipartite self-complementary graph, with respect to $K_{n,n}$. Then*

$$pm(G) = \int_0^{\infty} \rho(G;x)e^{-x}\ dx.\Box$$

Proof: See Godsil's paper for a similar expression for the usual sc-graphs.

1.63. Theorem[Godsil 1981]**.** *Let G be a bipartite self-complementary graph, with respect to $K_{n,n+a}$, for some $a \geq 0$. Define a new rook polynomial by*

$$\rho'(G;x) := \sum_{r=1}^{n} (-1)^r a_G(r)x^{n-r}.$$

Then

$$a_G(r) = \frac{1}{a!} \int_0^{\infty} \rho'(G;x)e^{-x}x^a\ dx, \text{ and}$$

$$\int_0^{\infty} \rho'(G;x)\rho'(K_{m,m+a};x)e^{-x}x^a\ dx = \begin{cases} n!(n+a)!a_G(n-m), & m < n \\ 0, & m \geq n.\Box \end{cases}$$

The Hadwiger and Strong Perfect Graph Conjectures

1.64. The clique number of a graph is the size of its largest complete subgraph. It is obvious that, if a graph contains a complete subgraph of size r, then its chromatic number will be at least r. The converse, however, does not hold. For example, C_{2k+1}, $k > 1$, has chromatic number 3 and clique number 2. The complement of such an odd circuit has clique number k but chromatic number $k+1$. (In general, the difference between clique number and chromatic number need not be 1 — it can be arbitrarily large [210]).

A graph is said to be *perfect* if for every induced subgraph the chromatic number equals the clique number. Obviously neither a perfect graph nor its complement can contain an induced odd circuit of size at least 5. Berge has conjectured that the converse is true.

Strong Perfect Graph Conjecture. *A graph G is perfect if and only if neither G nor $\overline{G}$ contains an induced odd circuit of size at least* 5.

1.65. Theorem[Corneil 1986]**.** *The Strong Perfect Graph Conjecture is true in general if and only if it is true for (biregular) self-complementary graphs.*

Proof: Obviously a self-complementary counter-example would show the conjecture to be false. We will show that, conversely, if the conjecture is false, there must be at least one (biregular) self-complementary counter-example.

Let G be a graph which contains no large odd holes but is not perfect. We create a sc-graph $\mathcal{P}(G,\overline{G})$ by forming the P_4-join of $(G,\overline{G},\overline{G},G)$ (see 1.26). $\mathcal{P}(G,\overline{G})$ is not perfect because it contains G as an induced subgraph. It can also be checked that $\mathcal{P}(G,\overline{G}) \cong \overline{\mathcal{P}(G,\overline{G})} = \mathcal{P}(\overline{G},G)$, and does not have any large odd hole, so it forms a counterexample to the SPGC.

To show the second part, we first construct a regular counterexample H. We let $r = \Delta(G)$ if $\Delta(G)$ is odd, and $r = \Delta(G)+1$ if $\Delta(G)$ is even. Construct the graph T_r as follows: take the complement of the union of one P_3 and $(r-1)/2$ copies of K_2; then add a K_2 to the middle vertex of the original P_3. T_r has degree sequence $(1, r, r, \ldots, r)$. Now to each vertex u in G we add $r - deg(u)$ copies of T_r, merging the end-vertices of the T_r's with u. This gives us a regular graph of degree r, which is imperfect because it contains G as induced subgraph, but which contains no large odd hole since neither G nor the T_r's do.

Then $\mathcal{P}(H,\overline{H})$ will be a biregular self-complementary counterexample to the SPGC. □

1.66. Although a graph with chromatic number r need not contain a clique of size r, Hadwiger conjectured that it must be contractible to a clique of size r. If true, this would be much stronger than the Four Colour Theorem.

Define a Hadwiger clique of G to be a collection of nonempty, pairwise disjoint subsets of $V(G)$ such that each set in the collection induces a connected graph and every two sets in the collection are joined with at least one edge. The Hadwiger number of G, $\eta(G)$, is the largest h such that G has a Hadwiger clique with h elements. Then the conjecture can be stated as follows:

Hadwiger Conjecture. *For any graph G, $\eta(G) \geq \chi(G)$.*

Zelinka conjectured the Nordhaus-Gaddum result $\eta(G) + \eta(\overline{G}) \leq n + 1$. This was apparently first proved [38] then disproved [226], but in any case Rao [302, Theorem 4.1] announced the following, without proof:

Theorem. *For a self-complementary graph of order n*

$$\chi(G) \leq \tfrac{n+1}{2} \leq \eta(G).$$

In particular, the Hadwiger conjecture is true for sc-graphs. □

The chromatic and total chromatic index

1.67. A proper edge-colouring is a colouring of the edges in which no two adjacent edges have the same colour. The minimum number of colours needed for a proper edge-colouring on G is called the chromatic index, $\chi'(G)$. It is well known [Vizing 1964] that

$$\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$$

where $\Delta(G)$ is the maximum valency of G. Graphs with $\chi'(G) = \Delta(G)$ are said to be of Class One, while those with $\chi'(G) = \Delta(G) + 1$ are of Class Two.

A proper total colouring is a colouring of the edges and vertices of G such that no two incident or adjacent elements have the same colour. The least number of colours with which this can be done is called the total chromatic index, $\chi^t(G)$.

Problem[Rao 1979a]**.** *Verify that the Behzad-Vizing conjecture holds for self-complementary graphs:*

$$\Delta(G)+1 \leq \chi^t(G) \leq \Delta(G)+2.$$

1.68. Wojda and Zwonek [391] have put forward the following:

Conjecture. *A self-complementary graph G is of Class Two if and only if it is regular.*

1.69. Now a graph G of order n is said to be *overfull* if $|E(G)| > \lfloor \frac{n}{2} \rfloor \Delta(G)$. Since in any colour class we may have at most $\lfloor \frac{n}{2} \rfloor$ edges, it is clear that overfull graphs are of Class Two. In particular, non-null regular graphs of odd order are of Class Two [34]. Wojda and Zwonek pointed out that in fact a self-complementary graph is overfull if and only if it is regular.

So the conjecture is easily verified in one direction, and there is some support for it in the other direction too. Chetwynd and Hilton made the following conjecture in [76]:

Conjecture. Let G be a graph of order n with the maximum vertex degree $\Delta(G) > \frac{n}{3}$. Then G is of Class Two if and only if G contains an overfull subgraph H with the maximum vertex degree $\Delta(H) = \Delta(G)$.

Wojda [390] proved that a graph G with at most $\frac{n(n-1)}{4}$ edges, and $\Delta(G) \geq \frac{n-1}{2}$, does not have any overfull subgraph H with $\Delta(H) = \Delta(G)$ and $|V(H)| < |V(G)|$. Therefore the Chetwynd-Hilton conjecture for sc-graphs is equivalent to the Wojda-Zwonek conjecture.

1.70. There is one other class of sc-graphs whose colour class is known — Wojda and Zwonek proved in [391] that every sc-graph with cyclic antimorphism is of Class One. Their proof essentially consists of Theorems 1.71 and 1.72, although it is complicated by numerous other considerations which, with hindsight, are seen to be unnecessary. We present the re-organised proof here for the sake of completeness. We note that Theorem 1.71, along with the observation in 1.69 on odd-order regular graphs, implies that a non-null connected circulant graph is of Class One if and only if it has even order. This was first proved by Sun [366] and other authors.

1.71. Theorem. *Let G be a circulant graph of even order with cyclic automorphism $\alpha = (1, 2, ..., 2k)$. Let G_1 and G_2 be the (isomorphic) subgraphs induced by the odd and even vertices respectively. If G_1 and G_2 are of Class One, or if there is an edge joining a vertex in G_1 to a vertex in G_2, then G is of Class One.*

Proof: Case 1: G_1 is of Class One.

Colour G_1 with $\Delta(G_1)$ colours, and denote the colour of each edge $e = \{v, w\} \in G_1$ by $c(e)$. The edges of G_2 can be written as $\alpha(e) = \{\alpha(v), \alpha(w)\}$, and we colour them by defining $c(\alpha(e)) = c(e)$. If there is any edge between G_1 and G_2, say $f = \{i, i+t\}$, for some odd t, then $f, \alpha^2(f), \alpha^4(f), \ldots, \alpha^{2k}(f)$ is a 1-factor of G which can be given some new colour $c(f)$. We repeat this process until all the 1-factors between G_1 and G_2 (if any) are coloured.
Case 2: G_1 is of Class Two.

Colour G_1 with $\Delta(G_1) + 1$ colours. Note that at each vertex v there are just $\Delta(G_1)$ incident edges, so there must be a colour which is missing at v.

By hypothesis, there is an edge between G_1 and G_2, say $f = \{i, i + t\}$, for some odd t. Then $f, \alpha^2(f), \alpha^4(f), \ldots, \alpha^{2k}(f)$ is a 1-factor F of G. We now colour the edges of G_2 by defining $c(\alpha^t(e)) = c(e)$. This ensures that for every edge $\alpha^{2j}(f) = \{i + 2j, i + 2j + t\}$, the colour missing at $i + 2j$ will be the same colour missing at $i + 2j + t$, and we can use this to colour $\alpha^{2j}(f)$. So now $H = G_1 \cup F \cup G_2$ is coloured with $\Delta(G_1) + 1 = \Delta(H)$ colours.

Any remaining edges between G_1 and G_2 can be grouped into 1-factors and coloured as before. □

1.72. Theorem. *If G is a self-complementary graph with cyclic antimorphism $\sigma = (1, 2, 3, ..., 4k)$ then G is of Class One.*

Proof: Consider the two subgraphs G_{odd} and G_{even} induced by the set of odd and even vertices, respectively. Any edge between G_{odd} and G_{even} must be of the form $e = \{i, i+t\}$, for some odd t. But since σ^2 is an automorphism of G, the edges $e, \alpha^2(e), \alpha^4(e), \ldots, \alpha^{2k}(e)$ will form a 1-factor of G. The edges between G_{odd} and G_{even} thus form some number s of 1-factors, which can be coloured with s colours.

Since σ^2 is an automorphism of G, G_{odd} will be a circulant graph of degree $r - s$, and G_{even} a circulant graph of degree $4k - 1 - r - s$ for some r. Without loss of generality we consider the case $r > 4k - 1 - r$. It is then enough to show that G_{odd} can be coloured with r colours, in other words, that it is of Class One.

Now G_{odd} has vertices $1, 3, 5, \ldots, 4k-1$. The subgraphs G_1, G_2 defined in Theorem 1.71 will have vertices $1, 5, 9, \ldots, 4k-3$ and $3, 7, 11, \ldots, 4k-1$ respectively. We will show that the vertex $1 \in G_1$ must be adjacent to at least one vertex of G_2 so that G_{odd} satisfies the conditions of Theorem 1.71.

Since σ is an antimorphism, σ^{1-2i} is also a antimorphism for any integer i. Now consider the pair of vertices $\{1, 2i\}$. We have

$$\{1, 2i\} \in E(G) \Leftrightarrow \{1+(1-2i), 2i+(1-2i)\} \notin E(G) \Leftrightarrow \{4k-2i+2, 1\} \notin E(G),$$

where labels are taken $\pmod{4k}$. So the vertex 1 must be adjacent to precisely half of the $2k$ vertices of G_{even} (in other words, $s = k$). If 1 is not adjacent to any of the vertices $3, 7, 11, \ldots, 4k-1$, then it can be adjacent to at most $k-1$ of the vertices of G_{odd}, and so its total degree is $r \leq 2k-1$, which contradicts the fact that $r > 4k-1-r$. □

1.73. There is a counterpart of 1.57 for the chromatic index, found by Vizing [378] and Alavi and Behzad [15]:

$$n - 1 \leq \chi'(G) + \chi'(\overline{G}) \leq 2(n-1) \text{ if } n \text{ is even, and}$$
$$n \leq \chi'(G) + \chi'(\overline{G}) \leq 2n - 3 \text{ if } n \text{ is even.}$$

For sc-graphs on $4k$ vertices this gives us the bounds

$$2k \leq \chi'(G) \leq 4k - 1$$

which is trivial since $2k \leq \Delta(G) \leq 4k-2$. For sc-graphs on $4k+1$ vertices we get the slightly more interesting

$$2k + 1 \leq \chi'(G) \leq 4k - 1.$$

The lower bound is only useful for odd order sc-graphs with $\Delta(G) = 2k$, that is, regular sc-graphs; but we already know from 1.69 that they are of Class Two. The upper bound at least tells us that odd order sc-graphs with end-vertices are of Class One.

More Nordhaus-Gaddum results

1.74. The Nordhaus-Gaddum results mentioned in 1.57 were only the first in a long and seemingly endless stream of results relating various graph parameters to those of the complementary graph. These give immediate corollaries for self-complementary graphs. Some of them, like those in 1.73, turn out to be of little, if any use. In any case, even if the original bounds are attained by an infinite number of graphs, this does not necessarily mean that the corresponding bounds for self-complementary graphs will be attained by any sc-graphs. We will therefore present only a sample of these results, omitting the (usually straightforward) derivation.

Let $\Psi(G)$ be the pseudoachromatic number, the greatest number of colours which can be used to colour the vertices such that for any two colours i, j, there exist adjacent vertices coloured i, j. (Note: there may even be adjacent vertices with the same colour). It follows from Gupta [167] that

$$\begin{aligned} \Psi(G) &\leq \tfrac{1}{2}\left\lceil \tfrac{4n}{3} \right\rceil \text{ and} \\ \chi(G) + \Psi(G) &\leq n + 1 \end{aligned}$$

1.75. A graph G is said to be k-degenerate if the minimum degree of each induced subgraph does not exceed r. Thus 0-degenerate graphs are null graphs and 1-degenerate graphs are forests. The *k-partition number ρ_k* of a graph is the least number of subsets in a partition of $V(G)$ such that each subset induces an k-degenerate subgraph. Obviously $\rho_0(G) = \chi(G)$, while the 1-partition number is also known as the vertex arboricity.

It follows from Lick and White [235] that for sc-graphs

$$\sqrt{\tfrac{n}{k+1}} \leq \rho_k(G) \leq \tfrac{n+1+2k}{2(k+1)}.$$

1.76. The k-clique chromatic number $\chi_k(G)$ is the smallest number of colours in a vertex-colouring of G in which no k-clique is monochromatic. Since $\chi_2(G) = \chi(G)$ we have another generalisation of the chromatic number. If we let R denote the Ramsey number $R(k, k)$, then Achuthan's [4] Nordhaus-Gaddum result gives us the following bounds for self-complementary graphs:

$$\sqrt{\tfrac{n}{R-1}} \leq \chi_k(G) \leq \tfrac{n+2k-3}{2(n-1)}$$

1.77. We say that a vertex v covers an edge e (and e covers v) if v is incident with e. The minimum number of vertices [edges] covering all the edges [non-isolated vertices] of G is called the vertex- [edge-] covering number of G and denoted by $\alpha_0(G)$ [$\alpha_1(G)$].

A set of vertices [edges] which are pairwise non-adjacent is said to be independent. The maximum size of an independent set of vertices [edges] is call the vertex- [edge-] independence number of G, denoted by $\beta_0(G)$ [$\beta_1(G)$]. For self-complementary graphs, it follows from Gallai [133] that

$$\alpha_0(G) + \beta_0(G) = n = \alpha_1(G) + \beta_1(G),$$

and from Cockayne and Lorimer [92] and Erdős and Schuster [116] that

$$\left\lfloor \frac{n+1}{3} \right\rfloor \leq \beta_1(G) \leq \frac{n}{2}.$$

1.78. An out-domination set of a digraph D is a set S of vertices such that every vertex of $D-S$ is adjacent *from* some vertex of S. The out-domination number $\gamma^+(D)$ is the minimum cardinality of an out-domination set. It follows from Chartrand, Harary and Quan Yue [70] that, for a connected self-converse digraph of order n

$$\gamma^+(D) \leq \frac{2n}{3}.$$

1.79. Structural Nordhaus-Gaddum results were investigated in a series of papers by Akiyama and Harary, with Ando, Exoo and Ostrand [7, 8, 9, 10, 11, 12, 13, 14]. Some other results can be seen in 1.55 and 2.22–2.23. For bipartite Nordhaus-Gaddum results we refer to Sivagurunathan and Mohanty [352] and Goddard, Henning and Swart [152].

1.80. Finally, we note a minor (no pun intended) result which may be useful. Let the vertices of G be labelled arbitrarily $v_1, v_2, \ldots, v_n$, and let M be the $n \times n$ matrix where

$$a'_{ij} = \begin{cases} deg(v_i), & \text{if } i = j, \\ 0, & \text{if } i \text{ is not adjacent to } j, \\ -1, & \text{if } i \text{ is adjacent to } j. \end{cases}$$

Then by the matrix-tree theorem (c.f. [387, Thm. 10C]), the cofactor of any element gives the number $T(G)$ of spanning trees of G. If, moreover, G is self-complementary, this means that

$$|M_1| = |(n-1)I + J - M_1| = T(G)$$

where M_1 is the matrix obtained from M by deleting the first row and column, I is the identity matrix and J the matrix with all entries 1.

Interestingly, when investigating spanning trees Sedláček [350] said that the only graphs he could find with the same number of spanning trees as their complement were self-complementary graphs. It would be too much to ask that this be true in general, but it would be nice to know how small a non-self-complementary example can be.

Chapter 2

Circuits, paths and cliques

2.1. It is part of most graph theorists' education to learn that if a graph on at least 6 vertices does not have a triangle, then its complement will. This immediately tells us that a self-complementary graph G on at least 6 vertices contains a triangle; in fact it follows from Albertson [18] that G must contain a triangle with two vertices of the same degree, and from Albertson and Berman [19] that G must contain a triangle such that the degrees of any pair of vertices differ by no more than 5.

We can look at these as results about triangles, circuits, complete graphs or Ramsey theory. In fact we will see that there are many results involving sc-graphs in each of these areas[1], but we start by considering triangles *per se*.

2.2. By Turán's theorem, all graphs with $m > \lfloor\frac{n}{2}\rfloor\lfloor\frac{n+1}{2}\rfloor$ edges contain a triangle, and this is best possible. But we can still say something about graphs with a few less edges. Brandt [47] has proved that every graph G that is not a star and does not contain C_3, C_4, C_5 or C_6 is isomorphic to a subgraph of $\overline{G}$. Any such graph with $\frac{1}{2}\binom{n}{2}$ edges would then be a self-complementary graph without a triangle, which is impossible for $n \geq 6$. So every graph on $n \geq 6$ vertices with $m \geq \frac{n(n-1)}{4}$ edges must contain a C_3, C_4, C_5 or C_6.

If Faudree *et al.*'s conjecture [121] is true, then Brandt's result can be improved to include just C_3 and C_4 in the conditions, which would lead to a

[1]There is a result about the strong perfect graph conjecture in 1.65.

similar improvement in the result above.

Triangle numbers and paths

2.3. The *triangle number* $t(v)$ $[t(e)]$ of a vertex [edge] in a graph G is the number of triangles in G containing v $[e]$, while $t(G)$ is the total number of triangles in G. The triangle number of a vertex v in $\overline{G}$ will be denoted by $\overline{t}(v)$. We use $N(v)$ $[E(v)]$ to mean the set of all vertices [edges] adjacent [incident] to v.

Obviously $t(v)$ is the size of the neighbourhood of v, i.e. the number of edges in the subgraph induced by $N(v)$; while $t(e)$ is the number of common neighbours of the end-vertices of e. See also 1.19.

We will need the following sc-graph $G_{\max}(n)$. Let $V(G_{\max}(4k)) = A \cup B$, where

$$A = \{1, 3, \ldots, 4k-1\}, B = \{2, 4, \ldots, 4k\},$$

A is complete and B is independent. Further, for $i \in A$ and $j \in B$, $i \sim j$ if and only if

- $j = i + 1$, or
- $j = i + 3$ and $i \equiv 3 \pmod 4$, or
- $j > i + 4$.

When $n = 4k + 1$ we add a vertex $4k + 1$ and join it to all the vertices of A.

Lemma[Nair 1994, Nair and Vijayakumar 1994]**.** *Let G be any graph. Then*

A. $t(v) = \frac{1}{2} \sum_{e \in E(v)} t(e)$ *for any* $v \in V(G)$.

B. $t(G) = \frac{1}{3} \sum_{v \in V(G)} t(v) = \frac{1}{3} \sum_{e \in E(G)} t(e)$. □

2.4. Theorem. *Let G be a graph with n vertices and m edges. Then*

A. *[Nair 1994, Nair and Vijayakumar 1994] For every* $v \in V(G)$

$$t(v) + \overline{t}(v) = \binom{n - d(v) - 1}{2} - m + \sum_{v \in N(v)} d(v).$$

B. [Goodman 1959] $t(G) + t(\overline{G}) = \binom{n}{3} - (n-1)m + \frac{1}{2}\sum_{v \in V(G)} d^2(v)$. $\square$

Proof: We will only prove A, as B then follows with little difficulty, as do the next two corollaries. Let $d := d(v)$ and $N := N(v)$, and define $\overline{N} := V(G) - N(v) - \{v\}$. Then $|N| = d$, $|\overline{N}| = n - d - 1$, $t(v)$ is the number of edges in $G[N]$, and $\overline{t}(v)$ the number of edges in $\overline{G}[\overline{N}]$.

Now let $D = \sum_{u \in N} d(u)$ and $\overline{D} = \sum_{u \in \overline{N}} d(u)$. Then $D + \overline{D} + d = \sum_{u \in V(G)} d(u) = 2m$.

The contribution to D by the d edges of G incident at v is d, and by the $t(v)$ edges in $G[N]$ is $2t(v)$. So the number of edges in G with one end in N and the other end in $\overline{N}$ is $D - d - 2t(v)$.

The number of edges in $G[\overline{N}]$ is $\binom{n-d-1}{2} - \overline{t}(v)$, and the contribution of these edges to $\overline{D}$ is $2\binom{n-d-1}{2} - 2\overline{t}(v)$. So the number of edges with one end in $\overline{N}$ and the other end in N is $\overline{D} - 2\binom{n-d-1}{2} + 2\overline{t}(v)$. Obviously, this must be equal to $D - d - 2t(v)$, and the result then follows. $\square$

2.5. Theorem. *If G is a self-complementary graph, then*

A. [Rao 1979c] The number of triangles depends only on the degree sequence. In fact

$$t(G) = \frac{1}{2}\sum_{i=1}^{n} \binom{d_i}{2} - \frac{n(n-1)(n-2)}{24}.$$

B. [Clapham 1973] The lower bound for A is achieved precisely by regular and almost regular graphs for n odd or even, respectively, and we have

$$t(G) = \begin{cases} \frac{1}{3}k(k-1)(4k-2) & \text{for almost regular sc-graphs} \\ & \text{on } 4k \text{ vertices,} \\ \frac{1}{3}k(k-1)(4k+1) & \text{for regular sc-graphs} \\ & \text{on } 4k+1 \text{ vertices.} \end{cases}$$

C. [Rao 1979b,c] The upper bound for A is achieved precisely by $G_{max}(n)$ (in fact there is no other sc-graph with the same degree sequence), and we have

$$t(G) = \begin{cases} \frac{k}{3}(k-1)(8k-1) & \text{if } G = G_{max}(4k), \\ \binom{2k}{2} + \frac{k}{3}(k-1)(8k-1) & \text{if } G = G_{max}(4k+1). \end{cases}$$

D. [Rao 1979b] There is a sc-graph G on $4k$ vertices with $t(G) = r$ if and only if r is even and lies between the upper and lower bounds given above.

E. [Rao 1979b] There is a sc-graph G on $n = 4k+1$ vertices with $t(G) = r$ if and only if r lies between the upper and lower bounds given above, with the exceptions of $r = 9, 12, 13$ for $n = 9$, and $r = 33, 41, 49, 54, 57$ for $n = 13$. □

2.6. We now consider the number of paths of length three or more. For convenience we define

$$f(n) = 2\binom{n(n-1)/4}{2} - 3\binom{n}{4}$$

and use $p_k(G)$ to denote the number of paths of length k in G.

Theorem[Rao 1979c]**.** *If G is a self-complementary graph with degree sequence $\pi = (d_1, ..., d_n)$, then*

$$p_3(G) = p_3(\pi) = f(n) + (n-5)\sum_{i=1}^{n}\binom{d_i}{2},$$

and so

$$p_3(G) - 2(n-5)t(G) = f(n) + \frac{n(n-1)(n-2)(n-5)}{12}.$$

The lower and upper bounds of $p_3(G)$ are thus achieved by the same graphs as in 2.5.B and 2.5.C, respectively, and we have

$$p_3(G) = \begin{cases} f(n) + (n-5)(8k^3 - 8k^2 + 2k) & \text{for almost regular sc-graphs on } 4k \text{ vertices,} \\ f(n) + (n-5)(8k^3 - 2k^2 - k) & \text{for regular sc-graphs on } 4k+1 \text{ vertices,} \\ f(n) + (n-5)\frac{2k}{3}(16k^2 - 15k + 2) & \text{if } G = G_{max}(4k), \\ f(n) + (n-5)\frac{k}{3}(32k^2 - 6k - 5) & \text{if } G = G_{max}(4k+1). \end{cases}$$ □

2.7. Theorem[Rao 1979c]**.** *If G is a s-c graph of order n, and $i > 3$, then*

$$p_i(G) \equiv (n-i)p_{i-1}(G) \pmod 2. \square$$

2.8. Corollary[Rao 1979c]**.** *If G is a sc-graph of order n, then*

A. $p_3(G) \equiv \lfloor \frac{n}{4} \rfloor \pmod 2$.

B. $p_4(G)$ *is odd if and only if* $n = 4k+1$ *and* k *is odd.*

C. $p_i(G)$ *is even for* $i \geq 5$*, and in particular*

D. [Camion 1975] the number of Hamiltonian paths is even iff $n > 5$. $\square$

2.9. Define the Hamiltonian path graph $H(G)$ of a graph G to be the graph having the same vertex set as G and in which two vertices u and v are adjacent if and only if G contains a Hamiltonian u-v path. A graph G is a self-Hamiltonian path graph if $G \cong H(G)$.

A graph G of even order $n \geq 4$ is chord additive if the vertices of G can be labelled so that

A. $v_1, v_2, \cdots, v_n, v_1$ is a Hamiltonian cycle C of G;

B. $d_G(v_i) = 2$ for each even i, $2 \leq i \leq n$;

C. C contains chords; and

D. $v_j v_k$ being a chord of C implies $v_{j+2h}v_{k+2h}$ is a chord of C for every integer h, where the subscripts are taken modulo n.

Chartrand, Kapoor and Nordhaus [71] showed that a Hamiltonian graph is a self-Hamiltonian path graph if and only if G is chord-additive or $G \in \{K_n, C_n, K_{m,m}, K_m + \overline{K}_m\}$; they conjectured that this result is true for all graphs. (In other words, they conjectured that every self-Hamiltonian path graph is Hamiltonian; we note that the trivial counterexample $\overline{K}_n$ should be excluded for this to hold). Thus no Hamiltonian self-complementary graph is a self-Hamiltonian path graph.

The question of whether there are non-Hamiltonian self-complementary path graphs is thus still open. We cannot use Camion's result on the parity of the number of Hamiltonian paths because, while a self-complementary self-Hamiltonian path graph would have to have at least $\frac{1}{2}\binom{n}{2}$ Hamiltonian paths, it may have more if there are two vertices joined by several Hamiltonian paths. What we need is a suitable upper bound.

2.10. Problem. *What is the maximum and minimum number of Hamiltonian paths and Hamiltonian circuits in a self-complementary graph of order n? [Rao 1979c] In particular, can there be $\frac{1}{2}\binom{n}{2}$ or more Hamiltonian paths?*

Circuits and Hamiltonicity

2.11. We now turn to circuits of any length, and then consider Hamiltonian paths and circuits in more detail. We state the results mostly without proof, but we outline a method of proof used repeatedly by Clapham and Rao, with good results. Let G be a self-complementary graph, and σ an antimorphism of G. Denote the cycles of σ by $\sigma_1, \ldots, \sigma_s$, with respective lengths $l_1, \ldots, l_s$, and the number of cycles by $s = s(\sigma)$. For the sake of definiteness we assume that the vertices of each cycle are numbered consecutively. We define the digraph $D = D(\sigma)$ to have vertex-set $\{1, \ldots, s\}$ and, for $i \neq j$, $i \to j$ if one of the following conditions hold:

(1) $l_i > 1$, $l_j > 1$ and some even vertex of σ_i is adjacent to some odd vertex of σ_j in G.

(2) $l_i = 1$ and the vertex of σ_i is adjacent to an odd vertex of σ_j.

(3) $l_j = 1$ and and an even vertex of σ_j is adjacent to the vertex of σ_i.

The following results then follow from this definition.

Lemma[Clapham 1973, 1974, c.f. Rao 1977a, 1979a]**.** *Let $D(\sigma)$ be as above. Then*

A. *For all $i \neq j$, either $i \to j$ or $j \to i$ or both.*

B. *Let $l_i > 1$ and $l_j > 1$. If $i \to j$, then all even vertices of σ_i are adjacent to the same (positive) number of odd vertices of σ_j, and all odd vertices of σ_j are adjacent to the same (positive) number of even vertices of σ_i.*

If $i \not\to j$, then every odd vertex of σ_i is adjacent to every even vertex of σ_j in G.

C. *If $l_i = 1$ and $i \to j$, then the vertex of σ_i is adjacent to all odd vertices and no even vertex of σ_j.*

D. *If $l_j = 1$ and $i \to j$, then the vertex of σ_j is adjacent to all even vertices and no odd vertex of σ_j.* □

2.12. It follows from A, and from the results of Rédei and Camion (2.13) that $D(\sigma)$ has a directed Hamiltonian path, and that if $D(\sigma)$ is strongly connected it has a directed Hamiltonian circuit (see Clapham [83] and Rao [305, Lemma 3.8], for further details). The proofs of many of the results in this section then have the following pattern.

Let G be a self-complementary graph for which we wish to prove property P, and let σ be an appropriate antimorphism, usually chosen to have $s(\sigma)$ as large as possible (for such a σ, l_i must be a power of 2 for all i). Prove that P holds when $s(\sigma) = 1$. When $s(\sigma) > 1$ order the cycles into the form $\sigma_a, \sigma_b, \ldots, \sigma_z$ where $a, b, \ldots, z$ is a Hamiltonian path or circuit in $D(\sigma)$, and use this to prove that property P holds. If $D(\sigma)$ is not strongly connected, then each of its strong components give a set of cycles of σ which induce a sc-subgraph of G; prove that P holds on each of these subgraphs, and then combine them in an appropriate way to show that it holds on G too.

The first result in our list is easily stated but impressive.

Theorem[Rao 1977a]**.** *If G is a self-complementary graph on $n > 5$ vertices, then for every integer $3 \le l \le n-2$, G contains a circuit of length l. Further, if G is Hamiltonian then it is pancyclic.* □

This means that the circumference of a self-complementary graph is either n (that is, the graph is Hamiltonian), $n-1$ or $n-2$. The graphs with circumference $n-2$ are scarce even among sc-graphs; see 7.16 for their enumeration.

2.13. Rao's theorem implies that every sc-graph has a path with at least $n-1$ vertices, but we can say more. Clapham and Camion both noted that Chvátal's theorem on Hamiltonian circuits has a simple corollary on Hamiltonian paths, which we apply below to sc-graphs.

Chvátal [79] showed that if G is a finite graph with $n \geq 3$ vertices, and degree sequence $d_1 \leq \cdots \leq d_n$ such that

$$d_i \leq i < \frac{n}{2} \Rightarrow d_{n-i} \geq n - i$$

then G has a Hamiltonian circuit.

Theorem[Clapham 1974, Camion 1975]**.** *Every self-complementary graph G has a Hamiltonian path.*

Proof: The degree sequence of G satisfies

$$d_i \leq i - 1 < \frac{n+1}{2} \Rightarrow d_{n+1-i} \geq n - i.$$

If we add a vertex v to G, and join it to all the vertices of G, we get a graph satisfying the conditions of Chvátal's theorem, and which must thus have a Hamiltonian circuit. Then G must contain a Hamiltonian path. □

Clapham also gave an independent proof of this result, establishing the method described in 2.12, while Camion proved that the number of Hamiltonian paths in a sc-graph is even iff $n \geq 5$.

2.14. Clapham then turned to infinite self-complementary graphs. Many theorems in infinite graph theory are concerned with locally finite graphs, that is, those graphs in which every vertex has finite valency; however, these can never be self-complementary, so Clapham considered instead *quasi-locally-finite* graphs, in which every vertex has either finite valency or covalency (the *covalency* of v in G is just the valency of v in $\overline{G}$).

Theorem[Clapham 1975]**.** *Every quasi-locally-finite sc-graph is countable, and has a spanning subgraph consisting of two* 1*-way infinite paths.* □

This result cannot be extended further, as Clapham gave an example of a countable self-complementary graph (not quasi-locally-finite) which requires a countable infinity of 1-way infinite paths to form a spanning subgraph.

2.15. Not every sc-graph has a Hamiltonian circuit, but Rao has given a good characterisation of those which do; this is in sharp contrast to the general situation, where the Hamiltonian problem remains one of the most difficult and important unsolved questions (it is in fact NP-complete — see 4.3).

A graph G is said to be *constricted* if there is a nonempty subset X of $V = V(G)$ such that $G[V - X]$ has more than $|X|$ components. The circuits C_n are not constricted, and thus no Hamiltonian graph is. For sc-graphs, the converse holds as well. In fact, Rao proved a stronger result.

G is said to be *highly constricted* if there is a nonempty subset X of V such that

A. $G[V - X]$ has more than $|X|$ components,

B. $G[X]$ is complete, and

C. for all $u \in X$ and $v \notin X$, $d_G(u) > d_G(v)$.

For convenience, we will denote the P_4-join of $(\overline{K}_k, K_k, K_k, \overline{K}_k)$ (see 1.26) by $G^*(4k)$. This graph is constricted but not highly constricted; Rao's result says that it is the only sc-graph of this type.

Theorem[Rao 1979d]**.** *A self-complementary graph G is non-Hamiltonian if and only if it is constricted. All graphs of this type are either highly constricted or isomorphic to $G^*(4k)$.* □

It also follows from [170] that if a sc-graph G is non-Hamiltonian and has minimum degree $\delta(G)$, then G contains a $K_{p,q}$ for all $p + q \leq \delta(G) + 1$.

2.16. We note that it is very easy to detect (algorithmically) whether a graph is isomorphic to $G^*(4k)$. Rao has shown that recognising highly-constricted sc-graphs is also very easy, as they are determined by their degree sequence. If a graph has degree sequence $\pi = (d_1, \ldots, d_n)$, then it is said to be a realisation of π, and we say that π is a graphic sequence. The sequence $\pi - f$ is defined to be $(d_1 - f, \ldots, d_n - f)$.

Theorem. *Let $G \neq G^*(4k)$ be a self-complementary graph with $n \geq 8$ vertices, and degree sequence $\pi = (d_1, \ldots, d_n)$ arranged in non-increasing order. Then the following are equivalent:*

A. G is Hamiltonian,

B. there is a Hamiltonian realisation (not necessarily self-complementary) of π, *C.* $\sum_{i=1}^{s} d_i < s(n-s-1)+\sum_{j=1}^{s} d_{n-j-1}$ *for every* $s < \frac{n}{2}$ *with* $d_s > d_{s+1}$. □

We note that the theorem is not valid for degree sequences in general [298], but it does tell us whether a potentially self-complementary degree sequence has a Hamiltonian realisation, in which case all self-complementary realisations are Hamiltonian or isomorphic to $G^*(4k)$.

In particular, every regular sc-graph is Hamiltonian. This is also true of biregular sc-graphs with degrees r, $4k-1-r$ where $k < r < 3k-1$. Maybe it is not a coincidence (see 4.16) that sc-graphs with cyclic antimorphisms have degrees r and $4k-1-r$ for some $k \leq r \leq 3k-1$.

2.17. Rao remarked that Chvátal's third conjecture in [80] is true for self-complementary graphs:

Conjecture. *If* G *is non-Hamiltonian, then it is degree majorised by a graph* H *containing a set* S *of vertices with* $|S| \leq \xi(G)$, $k(G-S) = |S|+1$.

Here, $k(G)$ denotes the number of components of G, and $\xi(G)$ its cyclability. A graph is *s-cyclable* if any s vertices lie on a common circuit; so "2-cyclable" is the same as "2-connected" (by Menger's theorem), while "n-cyclable" is the same as "Hamiltonian". The cyclability $\xi(G)$ is the largest s for which G is s-cyclable.

Finally, a graph H with degree sequence $(d'_1 \geq \cdots \geq d'_n)$ is said to degree-majorise a graph G with degree sequence $(d_1 \geq \cdots \geq d_n)$ if for all i, $d'_i \geq d_i$.

2.18. An r-factor of a graph is a regular spanning subgraph of degree r; obviously a Hamiltonian graph has a 2-factor, but so does G^*4k, for example. Rao [301] characterised sc-graphs with 2-factors, again in terms of their degree sequences.

Theorem. *Let* $G \neq G^*(4k)$ *be a self-complementary graph with degree sequence* π. *Then the following are equivalent:*

A. G *has a 2-factor*

B. $\pi - 2$ *is graphic*

C. π *has a realisation (possibly not self-complementary) with a 2-factor.* □

Rao conjectured that a sc-graph has an r-factor if and only if $\pi - r$ is graphic. Later [105] the following condition was added to the conjecture:

$$\sum_{i=1}^{s} d_i < s(n-f-1) + \sum_{j=1}^{s} d_{n-j-1} \text{ for every } s < \tfrac{n-4}{2} \text{ with } d_s > d_{s+1}.$$

However, Ando [25] constructed, for each r, a sc-graph with no r-factor, although its degree sequence π satisfies the condition above, and $\pi - r$ is graphic.

2.19. We now give a theorem, due to Rao [305, 308], which classifies sc-graphs according to their circumference, and characterises those sc-graphs which do not have a 2-factor. In fact, Rao deduced the previous theorems from the one below:

Theorem. *Let $G \neq G^*(4k)$ be a self-complementary graph on $4k+\epsilon$ vertices, where $\epsilon = 0$ or 1. If G is not Hamiltonian then $V(G)$ can be partitioned into two sets V_1, V_2, with $4k_1$, $4k_2+\epsilon$ vertices respectively, where $k_1 + k_2 = k$, such that*

A. $H_1 := G[V_1]$ and $H_2 := G[V_2]$ are sc-graphs.

B. $G[Hi] = K_{2k_1}$ and $G[Lo] = \overline{K}_{2k_1}$, where $Hi := \{v \in V_1 | d_{H_1}(v) \geq 2k_1\}$ and $Lo := \{v \in V_1 | d_{H_1}(v) < 2k_1\}$.

C. Every vertex of V_2 is adjacent to every vertex of Hi but to no vertex of Lo.

Moreover, G has circumference $n-2$ if and only if it satisfies A, B, C and:

D. $H_1 = G^*(4k_1)$.

G does not have a 2-factor if and only if it satisfies A, B, C and:

D'. If $k_1 > 1$ then H_2 does not have a 2-factor. □

2.20. Although the last few theorems have given complete characterisations, it is still useful to have conditions which are necessary but not sufficient for the existence of a 2-factor or Hamiltonian circuit. It is well-known that any graph has a Hamiltonian circuit if $d_v \geq \frac{n}{2}$ (Dirac's Theorem), or even if

$d_u + d_v \geq n$ for all non-adjacent vertices (Ore's Theorem c.f. [387, Thm. 7A]). Rao [305, Thm. A2] proved that for sc-graphs with $n \geq 8$ vertices, it is sufficient to have minimum degree at least $\frac{n}{4}$, or $d_u + d_v \geq \frac{n}{2}$ for all non-adjacent vertices; $G^*(4k)$ is the only exception here as it has no Hamiltonian circuit (though it does have a 2-factor).

Rao also constructed, for all feasible $n \geq 8$, sc-graphs with minimum degree $\lfloor \frac{n}{4} \rfloor - 1$, and thus $d_u + d_v \geq \frac{n}{2} - 2$ for all non-adjacent vertices, but which have no 2-factor, much less a Hamiltonian circuit. These graphs (see Figure 2.1) are obtained from $G^*(4k)$ by joining the vertices of the K_k's either to a single new vertex (for $n = 4k + 1$), or to all the vertices of a P_4 (for $n = 4k + 4$).

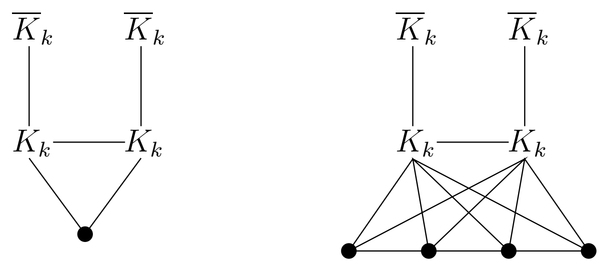


Figure 2.1: Sc-graphs with no 2-factor

2.21. A graph G in which every edge is contained in a Hamiltonian circuit is said to be *strongly Hamiltonian*. If, moreover, every pair of vertices are endpoints of a Hamiltonian path, then G is said to be *Hamiltonian connected*. (Contrast this with the concepts discussed in 2.9). A graph is *r-Hamiltonian* if removing any set of at most r vertices leaves a Hamiltonian graph. Rao [302] has posed the problem of characterising sc-graphs of each of the three types. Carrillo studied the first two concepts in [63, 64]; an example of his results is the following:

Theorem. *Let G be a self-complementary graph with cyclic antimorphism $(v_1\, v_2\, v_3\, \ldots\, v_{4k})$, where v_1 is adjacent to v_2, v_3, and either all or none of the vertices of the form v_{4s+2}, $s > 0$. Then G is strongly Hamiltonian if and only if it is Hamiltonian connected.*

2.22. Rao [309] has also looked at the line graphs of sc-graphs and obtained the following.

Theorem. *Let G be a sc-graph on $n \geq 8$ vertices. Then*

A. *$L(G)$ is pancyclic and strongly Hamiltonian.*

B. *$L(G)$ is Hamiltonian connected if and only if it is 1-Hamiltonian, if and only if the subgraph of G induced by the degree 2 vertices (if any) is* $\overline{K}_2$ or $\overline{K}_4$.

2.23. For further results about self-complementary line graphs see 1.55 and 5.2. The first part of 2.22.A also follows from a Nordhaus-Gaddum result of Nebeský [268] ('For any graph on more than 5 vertices, either $L(G)$ or $L(\overline{G})$ is pancyclic'). Similar corollaries follow from other work of Nebeský:

Theorem. *Let G be a self-complementary graph, and let* $\mathrm{eul}(G)$ *denote the size of the largest Eulerian subgraph of G.. Then*

A. *[270] The square of G is Hamiltonian connected.*

B. *[269] If G has end-vertices then* $\mathrm{eul}(G) = n-2$*; if G has no end-vertices, then* $\mathrm{eul}(G) \geq n-1$.

Lai [233] further proved that if G has no end-vertices and $n > 60$, then

(1) $\mathrm{eul}(G) = n$

(2) G has a 3-colorable cycle double cover, that is, G contains three subgraphs H_1, H_2, H_3, each of whose vertices have even degree (in H_i), such that every edge of G lies in exactly two H_i's.

The following problems remain open.

2.24. Problems[Rao 1979a]**.** *Characterise the following classes of graphs:*

A. *Strongly Hamiltonian sc-graphs*

B. *Hamiltonian-connected sc-graphs*

C. *r-Hamiltonian sc-graphs*

D. *Self-complementary digraphs with a Hamiltonian path*

E. Hamiltonian sc-digraphs

2.25. Conjecture[Rao 1979a]**.** *Every strongly connected sc-digraph has a Hamiltonian path.*

2.26. In the light of this conjecture, the following results are quite interesting:

Theorem[Wojda 1977]**.** *Let D be a self-complementary digraph. Then*

A. *The underlying graph of D contains a Hamiltonian path.*

B. *If D has an antimorphism in which all cycles have length at least* 4*, then D contains a* 2*-factor.* □

2.27. Theorem[Wojda 1977]**.** *Let σ be a permutation with a cycle of length* 1 *or* 2*, all other cycles having even length. Then there are sc-digraphs D_1, D_2 which both have σ as an antimorphism, but where D_1 has a* 2*-factor and D_2 does not.* □

Circuits and Hamiltonicity in sc-tournaments

2.28. For the sake of completeness we state some results which apply not just to sc-tournaments but to all tournaments in general. The best known is Rédei's Theorem [312]: every finite tournament has an odd number of directed Hamiltonian paths, and thus at least one (so Rao's conjecture above is at least true for sc-tournaments). Clapham [84] extended this to the infinite case: if T is a tournament in which every vertex has either finite in-valency or finite out-valency, then its vertices can be arranged in two 1-way sequences $\cdots \to v_3 \to v_2 \to v_1$ and $w_1 \to w_2 \to w_3 \to \cdots$ (where one of the sequences may be empty or finite).

A tournament contains a directed Hamiltonian circuit if and only if it is strongly connected [Camion 1959, Foulkes 1960]. A tournament on at least 9 vertices that is *not* strongly connected contains every non-directed Hamiltonian circuit [Havet 1998]. Every tournament contains every path

[Havet and Thomassé], with the exception of three sc-tournaments (the 3-cycle, the regular tournament on 5 vertices and the Paley tournament on 7 vertices) which do not contain antidirected Hamiltonian paths

$$v_1 \to v_2 \leftarrow v_3 \to \cdots \leftarrow v_n \text{ or } v_1 \leftarrow v_2 \to v_3 \leftarrow \cdots \to v_n.$$

The last two results were first proved by Thomason [369] for $n \geq 2^{128}$.

2.29. Salvi-Zagaglia [345] has studied directed circuits in sc-tournaments which are mapped on to their converse by an antimorphism. If σ is an antimorphism of a sc-tournament, and C [resp. a] is a directed circuit [arc] such that $\sigma(C) = C'$ [$\sigma(a) = a'$], then we say that C [resp. a] is σ-self-converse. The following is easy to prove:

Lemma. *If T is a sc-tournament with $n = 2k$ or $2k+1$ vertices, and σ an antimorphism of order 2, then there are exactly k σ self-converse edges.*
A σ-self-converse circuit of odd length must contain exactly one σ-self-converse arc and the fixed vertex of σ (which is impossible for $n = 2k$); while a σ-self-converse circuit of even length must contain exactly two σ-self-converse arcs, but not the fixed vertex of σ. □

After showing that every sc-tournament must have an antimorphism of order 2, Salvi-Zagaglia [346] then established a result which parallels Theorem 2.12.

2.30. Theorem. *Let T be a regular sc-tournament with antimorphism σ of order 2 and $2k+1$ vertices. Then every σ-self-converse arc a is contained in a σ-self-converse circuit of every length l, with the possible exception of $l = 4$ and either $l = 3$ or $l = 5$. In particular, T must contain at least k directed Hamiltonian circuits.*

Ramsey Numbers

2.31. The traditional Ramsey number $R(k,k)$ is the smallest n such that, for any graph G on n vertices, either G or $\overline{G}$ contains a K_k; equivalently, it is the least n such that any graph G on n vertices must contain either a K_k or a $\overline{K}_k$. It is convenient to define $n(k) = R(k,k) - 1$ to be the greatest integer

n for which there is a graph G on n vertices which does not contain a K_k or a $\overline{K}_k$.

Obviously, self-complementary graphs on at least $R(k,k)$ vertices must contain a K_k. So if we find a sc-graph on n vertices which does not contain a K_k, we must have $R(k,k) > n$. It is notoriously difficult to find bounds on the Ramsey numbers (much less exact values) but sc-graphs can and have been used with some success in this regard.

We note that the converse approach does not work — just because a sc-graph on n vertices contains a K_k does not mean that $R(k,k) \leq n$. This is quite easy to show — the P_4-join of $\overline{K}_k, K_k, K_k, \overline{K}_k$ (see 1.26) contains a K_{2k} and has just $4k$ vertices. But maybe if *all* sc-graphs on n vertices contain a K_k, then $R(k,k) \leq n$. The following conjecture is slightly stronger:

Conjecture[Chvátal, Erdős and Hedrlín 1972]**.** *Let $n^*(k)$ be the greatest n for which there exists at least one self-complementary graph on n vertices which does not contain a K_k. Then $n(k) = n^*(k)$.*

Chvátal *et al.* proved that $n^*(st) \geq (n^*(s) - 1)n(t)$, and in particular $n^*(2k) \geq 4n(k)$. They also noted that the conjecture is true for $k = 3$ and 4. In fact there is just one graph on $n^*(3)$ vertices which does not contain a K_3 or $\overline{K}_3$ and this graph is necessarily self-complementary. Kalbfleisch [216] proved that the same is true for $k = 4$.

The conjecture is strange, however, because it implies that $n(k) \equiv 0$ or 1 (mod 4) for all k. We can avoid this anomaly by using the almost self-complementary graphs [89, 106], which exist if and only if $n \equiv 2$ or 3 (mod 4). We note that, while for sc-graphs the statement "G does not contain a K_k or a $\overline{K}_k$" can be abbreviated to "G does not contain a K_k", this is not true for almost self-complementary graphs. We sum up all these observations in the following:

2.32. Problems. *Let $n^{**}(k)$ be the greatest n for which there is a self-complementary or almost self-complementary graph on n vertices which does not contain either a K_k or a $\overline{K}_k$. Are any of the following true?*

A. $n(k) = n^{**}(k)$;

B. The only graphs on $n^{**}(k)$ vertices which do not contain a K_k or a $\overline{K}_k$ are self-complementary or almost self-complementary;

C. *There is only one graph (up to complementation) on $n^{**}(k)$ vertices which does not contain a K_k or a $\overline{K}_k$ — either a unique self-complementary graph G, or else a unique almost self-complementary graph H and its complement $\overline{H}$.*

2.33. We now look at the concrete results obtained by using self-complementary graphs. It is possible, though, that there are some results on $R(k,k)$ not noted here which implicitly make use of self-complementary graphs.

Greenwood and Gleason [163] used Paley graphs to prove that $R(4,4) > 17$ (in fact, they showed that $R(4,4) = 18$). Burling and Reyner [56] used the same method to show that $R(6,6) > 101$ (which was already known), $R(7,7) > 109$, $R(8,8) > 281$ and $R(9,9) > 373$. Clapham [87] generalised the Paley graphs, and then used a computer to establish the known bound $R(5,5) > 41$, and the new bound $R(7,7) > 113$. Guldan and Tomasta [166] used Clapham's construction to show that $R(10,10) > 457$ and $R(11,11) > 521$, but improved this to $R(11,11) > 541$ by an even more general construction. (See 3.26–3.27).

In one of his earliest and best known papers [113] Erdős proved that $R(k,k) > (1+o(1))\frac{1}{e\sqrt{2}}k2^k/2$. The proof showed the power of the probabilistic method, and yet there were no known graphs which could be used to demonstrate this result explicitly. Constructive proofs, even of much weaker results, were still welcome; among these were the self-complementary graphs used by Rosenfeld [332] to show that $R(k,k) > ck^{\ln 5/\ln 2}$, and Abbott's [1] family of sc-graphs which showed that $R(k,k) \geq ck^{\ln 41/\ln 4}$ for some constant c. Chvátal, Erdős and Hedrlín [81] also constructed an infinite family of sc-graphs to demonstrate that $R(k,k) > 4\dot{2}^{(k-1)/4}$. The bound was later improved slightly to $R(k,k) > (1+o(1))\frac{\sqrt{2}}{e}k2^k/2$, and McDiarmid and Steger [248] managed to show that this bound could be established by a family of quasi-random regular self-complementary graphs. Meanwhile Rodl and Sinajova [328] showed, by a self-complementary construction, that $R(k,k) > 14\frac{(1+o(1)}{e\sqrt{2}}k2^{k/2}$.

2.34. We now see how the results of 2.1 can be generalised. A graph H is said to have the Ramsey repeated degree property if any graph G, or its complement, on at least n_H vertices must contain a copy of H with two vertices of equal degree in G, where n_H is some constant depending only on H. In particular, any self-complementary graph on at least n_H vertices must contain a copy of H with two vertices of the same degree.

The degree spread of a set of vertices X of a graph G is defined to be

$$\Delta_G(X) = \max_{x \in X} \deg_G(x) - \min_{x \in X} \deg_G(x).$$

The degree spread of a subgraph H of G is just the degree spread of $V(H)$.

With these definitions we can state Albertson and Berman's [18, 19] results as follows: the triangle has the Ramsey repeated degree property (where $n_{K_3} = 6$); and any sc-graph on at least 6 vertices must contain a triangle of degree spread at most 5. Albertson and Berman [19] showed that the first result could not be generalised to other complete graphs, because K_n does not have the Ramsey repeated degree property for $n \geq 4$. However, Erdős, Chen, Rousseau and Schelp [114] proved that all circuits and all bipartite graphs have the Ramsey repeated degree property.

They also showed that if we denote the Ramsey number $R(k, k)$ by r, then any sc-graph on at least $4(r-1)(r-2)$ vertices must contain a K_k with degree spread at most $r-2$. In particular, when $k = 3$ we see that any sc-graph on at least 80 vertices contains a triangle of degree spread at most 4, which improves Albertson and Berman's result. However, $R(k, k)$ increases rapidly, so that even for moderate values of k the result is not so impressive.

Šoltés [353] has shown that every graph H on at most 4 vertices (with the exception of K_1 and K_4) has the repeated degree property, and that in this case $n_H = R(H)$, where $R(H)$ is the least n such that any graph (or its complement) on n vertices contains a copy of H. He conjectured that in fact, $n_H = R(H)$ for all graphs. He also showed that the books B_k have the repeated degree property, where B_k consists of k triangles with a common edge.

Chapter 3

Regular sc-graphs

3.1. Any type of symmetry usually makes a structure more amenable to study, and more interesting. Self-complementarity is one type of symmetry which has attracted considerable attention; regularity is another symmetry which is even more intensely studied. It is therefore natural that regular self-complementary graphs should prove to be particularly interesting.

In what follows we often consider a graph G to be just a special type of digraph, obtained by replacing each edge of G by a pair of opposite arcs. For all digraphs, even those not obtained in this way, "regular" means that every vertex has both indegree and outdegree equal to some constant r.

There are various special classes of regular digraphs, such as the circulant and symmetric digraphs, which, under the additional condition of being self-complementary, fall into two hierarchies which intersect in the Paley digraphs on a prime number of vertices. Moreover, for digraphs on a prime number of vertices, one of the hierarchies collapses. These rich interconnections are displayed in Figure 3.3, and stated formally in 3.11 and 3.17, which are the keypoints of this chapter.

The presentation of any material, and the order it is presented in, is always subjective, but even more so in this chapter. A historical presentation, for example, would give Kotzig's famous problems [227] centre stage, as they provided the motivation for much of the work that was done [122, 198, 263, 265, 306, 334, 393]; while the hierarchies we are focusing on emerged piecemeal as a byproduct of this and other work (mainly by Hong Zhang [407, 408, 409]). We chose instead to present Kotzig's problems as a catalyst (which is probably how they were meant), and the hierarchies as the end result. One reason for this is that, up to now, the picture has never been presented as a whole but only seen in bits and pieces.

3.2. First, some basic results on regular and almost regular sc-graphs, and pointers to some other results scattered throughout this thesis. An almost regular graph is one which has exactly two degrees, s and $s+1$, for some s.

A. A regular self-complementary graph G must have $4k+1$ vertices and degree $2k$ for some k, and diameter 2. An almost regular sc-graph H must have $4k$ vertices, of which half have degree $2k$ and half $2k-1$, for some k. Moreover, the regular and almost regular sc-graphs are in one-one correspondence (see 1.41).

B. Every regular or almost regular sc-graph, apart from P_4, is Hamiltonian (see 2.16).

C. Every antimorphism of a self-complementary graph is also the antimorphism of a regular or almost regular sc-graph (see 4.17).

D. The sc-graphs with the least number of triangles, and the least number of P_4's are just the regular and almost regular sc-graphs (see 2.5–2.6).

See 1.53–1.54 for results on the eigenvalues of regular and circulant self-complementary graphs; 1.68–1.69 for the chromatic index of rsc-graphs; 4.10 for applications of rsc-graphs to the isomorphism problem; and 7.18 to 7.25 for enumeration of vertex-transitive self-complementary graphs and digraphs.

Two interlocking hierarchies

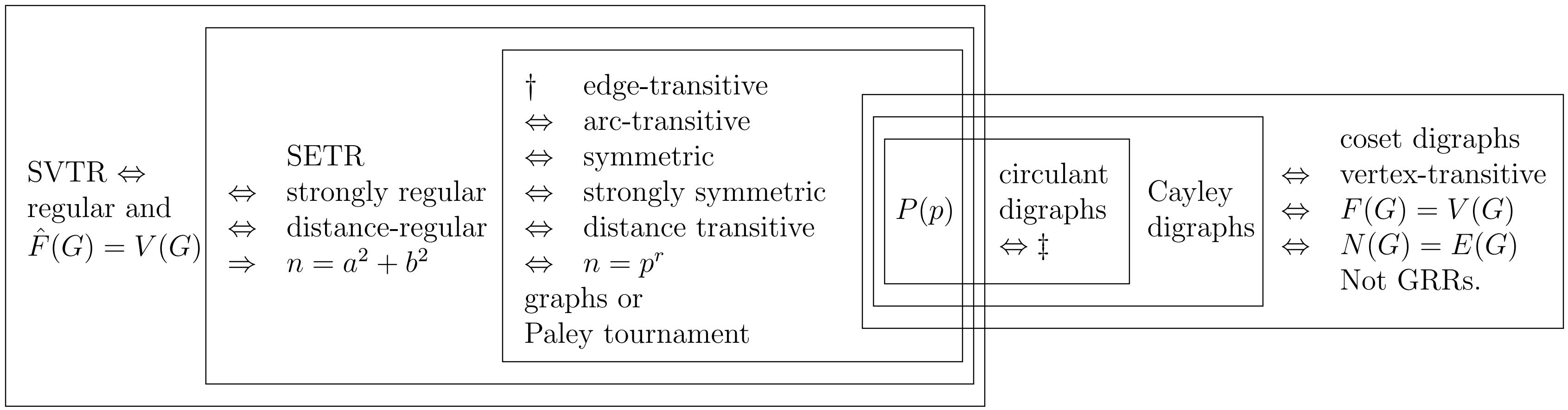
3.3. We now start working our way towards the first hierarchy. The presentation is necessarily heavy with definitions to introduce the concepts along the way.

A graph G is vertex-transitive if for any two vertices u, v there is an automorphism mapping u onto v. It is edge-transitive if, for any two edges ab, xy, there is either

A. an automorphism α such that $\alpha(a) = x$, $\alpha(b) = y$, or

B. an automorphism β such that $\beta(a) = y$, $\beta(b) = x$.

We say that G is arc-transitive if, for any two edges, both A and B occur.



Legend	
SVTR	Strongly vertex triangle regular
SETR	Strongly edge triangular regular
$P(p)$	Paley graph or tournament on p vertices, p prime
GRR	Graphical Regular Representation
†	for all graphs, arc tr. ⇔ s. symm. for digraphs, edge transitive and symmetric are not defined
‡	$p \mid n \Rightarrow p \equiv 1 \pmod 4$

Figure 3.1: Regular self-complementary graphs — two interlocking hierarchies

A graph which is both vertex- and edge-transitive is said to be symmetric, whereas a graph which vertex- and arc-transitive is said to be strongly symmetric.

Similar definitions hold for digraphs, except that we can only define arc-transitive digraphs, not edge-transitive ones; and so we can use the terms "symmetric digraph" and "strongly symmetric digraph" interchangeably.

3.4. Proposition. *A connected edge-transitive graph is either vertex-transitive or bipartite.*

Proof: Let uv be any edge of a connected edge-transitive graph G. All vertices are incident to some edge, and must thus be in the orbit of u or v. If the two orbits are identical, G is vertex-transitive. If not, then we cannot have any edge xy where x and y are in the same orbit, since no automorphism would map uv onto xy. So G is bipartite. $\square$

3.5. Proposition. *Let $D \neq \vec{K}_2$ be a connected arc-transitive digraph. Then D is vertex-transitive.*

Proof: If D has a source v, then there is an arc vw for some w. But since for any other arc xy there is an automorphism α such that $\alpha(v) = x$, $\alpha(w) = y$, the tail of every arc must be a source. So $D = s\vec{K}_2 \cup tK_1$, that is, s disjoint copies of $\vec{K}_2$ and t isolated vertices. This is disconnected unless $D = K_1$ or $\vec{K}_2$ (the same argument works if D has a sink). Both are arc-transitive and self-complementary, but K_1 is vertex-transitive while $\vec{K}_2$ is not.

If D is any other arc-transitive self-complementary graph, any two vertices a, b, are the tails of some arcs aa', bb', and by arc transitivity there is an automorphism mapping a to b. $\square$

3.6. Self-complementary graphs and digraphs are connected, and the only sc-graph which is bipartite is P_4, which is not edge-transitive. (We are not talking about bipartite self-complementary graphs, which is a different concept altogether). So we have proved that, in Theorem 3.11, A $\Leftrightarrow$ C and B $\Leftrightarrow$ D.

Zhang [407] used group theoretic arguments to prove that a symmetric sc-graph must be strongly symmetric, so that conditions A through D are all equivalent. In [409] he proved the following:

Lemma. *Let D be a symmetric sc-digraph on n vertices. Then*

A. if $n \equiv 1 \pmod 4$, D is just a graph, and

B. if $n \equiv 3 \pmod 4$, D is a tournament.

Proof: By arc transitivity either all vertices are joined by arcs in both directions, or none are. That is, D is either a sc-graph or an oriented sc-graph, and in the latter case it is a tournament since $|E(D)| = \frac{n(n-1)}{2}$.

If $n \equiv 3 \pmod 4$, $|E(D)|$ is odd so D cannot be a graph.

If $n \equiv 1 \pmod 4$, $|E(D)|$ is even, so its automorphism group has even order, by arc-transitivity and the Orbit-Stabiliser theorem. But the automorphism groups of tournaments have odd order, so the result follows. $\square$

Zhang also proved that symmetric sc-digraphs exist if and only if $n = p^r$ for some odd prime p; and, in particular, the only symmetric sc-tournaments are the Paley tournaments. Sufficiency is proved by the existence of Paley graphs and tournaments, necessity by his algebraic characterisation of the symmetric sc-digraphs. (In fact [37, 217], the Paley tournaments are the only arc-transitive tournaments).

We defer the definition of Paley graphs and tournaments to 3.18, and Zhang's characterisation to 3.29, in order not to interrupt the flow of arguments.

In [408] Zhang proved that the only circulant graphs G for which both G and $\overline{G}$ are edge-transitive are mK_n, $K_{n,n,\ldots,n}$, and the Paley graphs on a prime number of vertices, of which only the latter are self-complementary. Similar results hold for digraphs.

Chao and Wells [67] showed that if $n = p$ is prime there is a non-null symmetric digraph of degree r if and only if $r|p-1$, and that this digraph is unique. For $r = \frac{p-1}{2}$ we get the Paley graph or Paley tournament.

This bevy of remarkable results establishes the statements in the middle of Theorem 3.11, and we now turn to the task of showing that B $\Leftrightarrow$ E $\Rightarrow$ H $\Leftrightarrow$ F.

3.7. Let $d(u,v)$ denote the distance between vertices u and v, and define $N_i(u) := \{v|d(u,v) = i\}$. A connected graph G is *distance regular* if, for any two vertices u, v, and any two integers i,j, $|N_i(u) \cap N_j(v)|$ depends only on $d(u,v)$. Since, irrespective of $d(u,v)$, $N(u) = \cup_i N_i(u) \cap N_j(v)$, G is regular.

A graph G is said to be *distance transitive* if, for any vertices u, v, u', v' with $d(u,v) = d(u',v')$, there is an automorphism mapping u to v and u' to v'. If G is connected it is obviously distance regular, and by considering the case when $d(u,v) = 1$ we see that G is arc-transitive. So B $\Leftarrow$ E $\Rightarrow$ H.

A strongly regular graph with parameters (n, r, λ, μ) is a regular graph of order n and degree r, where every pair of adjacent [non-adjacent] vertices has λ [μ] common neighbours. We abbreviate "strongly regular self-complementary graph" to srsc-graph. It is well-known, and easy to check, that $\overline{G}$ will also be strongly regular with parameters $(n, n-r-1, n-2r+\mu-2, n-2r+\lambda)$.

Lemma. *A self-complementary graph is distance transitive if and only if it is arc-transitive, and distance regular if and only if it is strongly regular.*

Proof: We have already seen that all distance-transitive graphs are arc-transitive. Now, let G be arc-transitive and self-complementary, and let u, v, u', v' be any vertices. If $d(u,v) = d(u',v') = 1$, the required automorphism exists by arc-transitivity. If $d(u,v) = d(u',v') > 1$ then uv and $u'v'$ are edges of $\overline{G}$ which is also arc-transitive, so there is an automorphism of $\overline{G}$ (and thus of G) mapping u to u' and v to v'.

For a sc-graph G to be either distance regular or strongly regular, it must at least be regular and thus, by 3.2.A, have diameter 2. But it follows from the definitions that any graph H of diameter 2 is distance regular if and only if it is strongly regular. $\square$

Results on the properties of strongly regular sc-graphs are given in Proposition 3.32. For now, we will restrict ourselves to the fact that they can only exist when n is the sum of two squares [351]. We know that they do exist when n is a prime power [157, 351]; this also follows from the existence of the Paley graphs and other symmetric sc-graphs.

3.8. For any graph G, and any vertex [edge] v [e], we define $t(v)$ [$t(e)$] to be the number of triangles of G containing v [e]. We also define $\overline{t}(v)$ to be the number of triangles of $\overline{G}$ which contain v.

A *vertex triangle regular* [*edge triangle regular*] graph is one in which $t(v)$ [$t(e)$] is the same for all vertices [edges]. If, moreover, the graph is regular, then we say that it is *strongly vertex triangle regular* [*strongly edge triangle regular*], or just SETR [SVTR] for short.

Theorem[Nair 1994, Nair and Vijayakumar 1996]**.** *A graph G is strongly regular if and only if both G and $\overline{G}$ are strongly edge triangle regular.*

Proof: If G is strongly regular with parameters (n, r, λ, μ) then it is also SETR with degree r and $t(e) = \lambda$, and $\overline{G}$ is SETR with degree $n - r - 1$ and $t(e) = n - 2r + \mu - 2$.

Conversely, let G and $\overline{G}$ be strongly edge triangle regular with $t(e) = t$ and $t(e) = \overline{t}$, respectively, and let G have degree r. Then G is strongly regular with parameters $(n, r, t, 2r + \overline{t} - n + 2)$. □

3.9. So conditions F and G of Theorem 3.11 are equivalent. The previous theorem also gives an alternative proof of the fact that edge-transitive sc-graphs are strongly regular. We now show that G $\Rightarrow$ I $\Leftrightarrow$ J. We denote the set of edges incident to a vertex u by $E(u)$.

Lemma[Nair 1994, Nair and Vijayakumar 1996]**.** *If G is SETR then it is also SVTR. And if G is SVTR, then so is $\overline{G}$.*

Proof: If G has degree r and $t(e) = t$ for every edge e then, for any vertex u, $t(u) = \frac{1}{2}\sum_{e\in E(u)} t(e) = \frac{1}{2}rt$.

If G has degree r and $t(v) = t'$ for any vertex v, then its complement has degree $n - r - 1$ and, by 2.4.A, $\overline{t}(v) = \binom{n-r-1}{2} - \frac{nr}{2} + r^2 - t'$ for all vertices. □

We note that Nair gave examples of a graph which is vertex triangle regular but not SVTR, one which is SVTR but not SETR, and a graph which is SETR but whose complement is not SETR.

3.10. For any graph G we define the set $\hat{F}(G)$ to be

$$\hat{F}(G) := \{u \in V(G) | t(u) = \overline{t}(u)\}.$$

Lemma. *Let G be a regular sc-graph of order $4k + 1$, and σ an arbitrary antimorphism. Then*

A. *For any vertex v, $t(v) + t(\sigma(v)) = t(v) + \overline{t}(v) = 2k(k - 1)$.*

B. *$F(G) \subseteq \hat{F}(G) = \{u \in V(G) | t(u) = k(k - 1)\}$.*

C. *$\hat{F}(G) = V(G)$ if and only if G is strongly vertex triangle regular.*

Proof: Part A follows from 2.4.A. Part B is an obvious consequence. Now $F(G) \neq \emptyset \Rightarrow \hat{F}(G) \neq \emptyset$ and part C then follows. □

A historical note: the set $\hat{F}(G)$ was originally defined by Kotzig [227] only for regular self-complementary graphs, with the defining propery given in B above. Kotzig later communicated the result in part A to Rosenberg [331], but it was Nair and Vijayakumar [263, 264] who introduced the definition which we gave here, and which is valid for all graphs. Nair and Vijayakumar also noted that for all graphs, $\hat{F}(G) = \hat{F}(\overline{G})$. Kotzig knew that $F(G) \subseteq \hat{F}(G)$ for regular self-complementary graphs; with the new definition, this inclusion is obviously valid for all sc-graphs.

The first hierarchy has now been completely established and we can state it in full.

3.11. The First Hierarchy. *Let $G \neq \vec{K}_2$ be a self-complementary graph or digraph. Then the following conditions are equivalent:*

A. G is edge-transitive

B. G is arc-transitive

C. G is symmetric

D. G is strongly symmetric

E. G is distance transitive

A symmetric sc-digraph D on n vertices exists if and only if $n = p^r$ for some odd prime p; and D is either a graph (whenever $n \equiv 1 \pmod 4$) or the Paley tournament (whenever $n \equiv 3 \pmod 4$).

The Paley digraphs on a prime number p of vertices are the only symmetric sc-digraphs on p vertices; they are also the only circulant symmetric sc-digraphs.

For sc-graphs, either of conditions A, B, C, D, E imply the following equivalent conditions:

F. G is strongly regular

G. G is strongly edge triangle regular

H. G is distance regular

and each of these, in turn, implies the following equivalent conditions:

I. *G is strongly vertex triangle regular.*

J. *G is regular and $\hat{F}(G) = V(G)$.*

Strongly regular sc-graphs on n vertices exist when $n \equiv 1 \pmod 4$ is a prime power; they do not exist when n is not the sum of two squares. □

3.12. We now turn to the second hierarchy, much of which is valid for all graphs and digraphs, not just the self-complementary ones.

If the vertices of a digraph D can be labelled $1, 2, \ldots, n$ so that $(1, 2, \ldots, n)$ is an automorphism of D, we say that D is *circulant*. Evidently all circulant graphs are vertex-transitive, but not *vice versa*. Bridging the gap between these two classes are the Cayley digraphs.

Let Γ be a group with unit element 1. Let S be a subset of Γ such that

A. $1 \notin S$.

The *Cayley digraph Cay*(Γ, S) has vertex set Γ and edge set $\{(g, gs) : \ g \in \Gamma,\ s \in S\}$. In other words, there is an arc (a, b) iff $a^{-1}b \in S$. We need condition A to exclude loops. If we add conditions B or C we get, respectively, Cayley graphs or connected Cayley digraphs.

B. $s \in S \Leftrightarrow s^{-1} \in S$.

C. S generates Γ.

If instead of B we have

B1 $s \in S \Leftrightarrow s^{-1} \notin S$

we get oriented Cayley graphs, which are tournaments iff $|S| = \frac{|\Gamma|-1}{2}$.

Now circulant digraphs are precisely the Cayley digraphs of cyclic groups, that is, G is circulant iff $G \cong \mathrm{Cay}(Z_n, S)$ for some $S \subseteq Z_n$; S is sometimes called the symbol. Moreover, all Cayley digraphs are vertex-transitive[1], but not all vertex transitive digraphs are Cayley digraphs. We therefore define a more general type of digraph.

Let Γ be a finite group, and $H \leq \Gamma$. Let S be a subset of Γ such that

[1]The left-translation $\lambda_{ba^{-1}} : g \to ba^{-1}g$ will map a to b. Note that $\lambda_{ba^{-1}}$ is an isomorphism of the digraph $\mathrm{Cay}(\Gamma, S)$ but not a group isomorphism of Γ.

A'. $S \subseteq \Gamma - H$.

The coset digraph $Cos(\Gamma, H, S)$ has as vertex set the left cosets of H in Γ, and arc set $\{(gH, gsH : \ g \in \Gamma, \ s \in S\}$. In other words, there is an arc (aH, bH) iff $a^{-1}b \in HSH$. Equivalently, aH is adjacent to bH iff there are $x \in aH$, $y \in bH$ such that $x^{-1}y \in S$.

As before, condition A' ensures we get no loops, while adding condition B' (or B1 above) or C' will give us coset graphs (or oriented coset graphs) or connected coset digraphs.

B'. $s \in S \Leftrightarrow s^{-1} \in S$

C'. $H \cup S$ generates Γ.

Besides, when $H = \{1\}$ the digraph $Cos(\Gamma, H, S)$ is just the Cayley digraph $\mathrm{Cay}(\Gamma, S)$.

3.13. Sabidussi [340] has proved that a digraph G is vertex-transitive if and only if it is isomorphic to some coset digraph. Thus circulant digraphs are a subclass of Cayley digraphs, which are in turn a subclass of coset digraphs, which are just the vertex-transitive digraphs.

If G is vertex-transitive and for any two vertices v, w, there is just one automorphism α mapping v to w, then G is said to be a *Graphical Regular Representation* (or GRR) of its automorphism group. Now if G is also self-complementary, then there is some antimorphism σ fixing v, and then $\alpha\sigma^2 \neq \alpha$ also maps v to w. So [236] no sc-graph is a GRR.

Turner [373] proved that a digraph on a prime number of vertices is vertex-transitive if and only if it is circulant — if $Aut(D)$ acts transitively on $V(D)$ then, by the Orbit-Stabilizer Theorem, n divides $|Aut(D)|$; if, moreover, $n = p$ is prime, then $Aut(D)$ must contain an n-cycle, that is, a cyclic automorphism. (Incidentally, there cannot be non-null disconnected vertex-transitive digraphs on p vertices, as the order of the components would have to divide p). So the hierarchy collapses.

3.14. We note that so far we have not made any use of self-complementarity. When we do add this condition, then we can say a lot more. The equivalence of conditions A, B, C of 3.17 was established by Rao [306] for self-complementary graphs, and Robinson's theorem [324, Thm. 2] allows us to extend this to sc-digraphs (see 1.34, 1.35 and 5.7).

Zhang [408] showed that Paley digraphs are the only symmetric circulant sc-digraphs. Chia and Lim [78] proved that every vertex-transitive (that is, circulant) sc-digraph on a prime number p of vertices is either a graph or a tournament. Klin, Liskovets and Pöschel [222] showed, by counting arguments, that when p is a prime congruent to 3 (mod 4), every circulant sc-digraph on p^2 vertices is a tournament. In the light of Zhang's analogous result for symmetric sc-digraphs [409], it would be interesting to know whether there are any circulant sc-digraphs that are neither graphs nor tournaments.

Alspach [20] gave an elementary proof that every circulant digraph is self-converse — if $(v_0, v_1, \ldots, v_{2k})$ is a circulant automorphism of D, then it can easily be checked that $\sigma : v_i \mapsto v_{2k+1-i}$ maps D to D'. In particular, every circulant tournament is self-complementary, and every vertex-transitive tournament on a prime number of vertices is both circulant and self-complementary. Alspach gave an example of a tournament on 21 vertices that is vertex-transitive but not circulant; however, he showed that it is not self-complementary either, and this led him to ask whether there are any non-circulant vertex-transitive sc-tournaments (Problem 3.46.C).

3.15. Now circulant (and thus self-complementary) tournaments exist for all odd n, but the situation with respect to graphs is very different. Sachs [341] showed by construction that circulant sc-graphs exist for all n whose prime divisors are all congruent to 1 (mod 4). He suspected that the converse was true, and used a detailed investigation of circulant matrices and their eigenvalues to establish non-existence for

- $n = pq$, where p and q are two distinct primes, both congruent to 3 (mod 4);
- $n = p^{2s}$ for any integer s, where p is a prime congruent to 3 (mod 4) (Klin, Liskovets and Pöschel [222] obtained this non-existence result just for the case $n = p^2$, using counting arguments);
- $n = 3^2 p$ where p is a prime congruent to 1 (mod 4), $p > 5$.

Sachs' hunch was proved in full generality by Fronček, Rosa and Širáň [129] using group- and graph-theoretic techniques. They also showed that in the case $n = pq$ treated by Sachs, not even Cayley self-complementary graphs can exist; this is just a special case of a result by Muzychuk (see 3.16).

Suprunenko [367] constructed a particular class of Cayley sc-graphs using group automorphisms. A group Γ is *reflexible* if there is a partition $\Gamma - \{1\} = \Omega \cup \overline{\Omega}$ into two disjoint subsets such that

A. $\Omega = -\Omega$ and $\overline{\Omega} = -\overline{\Omega}$

B. there is a group automorphism $\phi : \Gamma \to \Gamma$ such that $\phi(\Omega) = \overline{\Omega}$.

Such an automorphism automatically induces a graph isomorphism from $\mathrm{Cay}(\Gamma, \Omega)$ to $\mathrm{Cay}(\Gamma, \overline{\Gamma})$, so that a reflexible group gives us a Cayley sc-graph. It is not known whether the converse is true, but Suprunenko characterised all Abelian reflexible groups, showing that none exist for the case $n = pq$ mentioned above; of course, this is just a special case of Fronček, Rosa and Širáň's result.

If we omit condition A above, we get a *semi-reflexible* group which gives rise to a Cayley sc-digraph. If, moreover, $\Omega = -\overline{\Omega}$, we get an *anti-reflexible* group, which produces a Cayley sc-tournament.

We note that Figueroa and Giudici [123] also described a group-theoretic method of constructing vertex-transitive self-complementary graphs, except that they used transitive permutation groups.

3.16. Zelinka [406] asked whether there are vertex-transitive sc-graphs for each $n \equiv 1 \pmod 4$, and constructed examples when n is a prime for which 2 is a primitive root of the Galois field $\mathrm{GF}(n)$; that is, for any m, $1 \leq m \leq \frac{n-1}{2}$, 2^m is congruent to neither 1 nor $-1 \pmod n$. Zelinka also constructed infinite vtsc-graphs of countable order and of order the power of the continuum.

Of course, Sachs' result mentioned above is much more general for the finite case, as it shows that vertex-transitive (and even circulant) sc-graphs exist whenever every prime divider of n is congruent to $1 \pmod 4$. Rao [306] gave an even better result, which we can adapt to give an alternative proof of Sachs' result.

The composition $G(H)$ of two graphs G, H, consists of replacing every vertex of $V(G)$ by a copy of H, and replacing every edge $uv \in E(G)$ by a bundle of edges joining the two corresponding copies of H. This is just a special case of the generalised G-join defined in 1.26, and it is easy to see that if G and H are both self-complementary [or both vertex-transitive], then $G(H)$ is also self-complementary [or vertex-transitive].

Now the Paley graphs are a well-known family of vertex-transitive self-complementary graphs, and they exist whenever $n = p^r \equiv 1 \pmod 4$, for some prime p (see 3.18). Repeated composition of Paley graphs shows their existence for all multiples of such numbers; Muzychuk [**?**] showed that they do not exist for any other values of n.

Theorem[Rao 1985, Muzychuk 1999]**.** *Let $p_1^{r_1} p_2^{r_2} \cdots p_s^{r_s}$ be the prime factor decomposition of n. There exists a vertex-transitive self-complementary graph of order n if and only if $p_i^{r_i} \equiv 1 \pmod 4$ for every i, $1 \leq i \leq s$.* $\square$

This theorem is equivalent to saying that a vertex-transitive sc-graph on n vertices exists if and only if n is the sum of two squares [57, Ex. 6.34]. We also know that strongly regular sc-graphs can *only* exist for these values of n [351], and apparently not even all of them, though they do exist when n is a prime power.

We can use a very similar method to establish Sachs' result on circulant graphs. The Paley graphs on a prime number of vertices are circulant. Now it is not difficult to see that if G and H are both circulant, then so is $G(H)$. For let $(v_1 v_2 \dots v_m)$ and $(w_1 w_2 \dots w_n)$ be circulant automorphisms of G and H respectively; then

$$((v_1, w_1) \dots (v_m, w_1)(v_1, w_2) \dots (v_m, w_2)(v_1, w_3) \cdots (v_1, w_n) \dots (v_m, w_n))$$

is a circulant automorphism of $G(H)$. So if $n = p_1^{r_1} p_2^{r_2} \cdots p_s^{r_s}$, where each p_i is a prime congruent to 1 (mod 4), then the composition of r_1 $P(p_1)$'s, r_2 $P(p_2)$'s and so on (in any order) gives us a circulant sc-graph on n vertices.

3.17. The Second Hierarchy. *Every circulant digraph is a self-converse Cayley digraph, which in turn is a vertex-transitive (or coset) digraph. If G is a self-complementary graph or digraph, then it is not a GRR and the following are equivalent:*

A. *G is vertex-transitive.*

B. *$F(G) = V(G)$.*

C. *$N(G) = E(G)$.*

For digraphs D on a prime number p of vertices this hierarchy collapses and, moreover, consists just of graphs and tournaments (the vertex-transitive sc-graphs and all vertex-transitive tournaments). On p^2 vertices (where p is prime, $p \equiv 3 \pmod 4$) every circulant sc-digraph is a tournament.

Circulant sc-tournaments are just the circulant tournaments, which exist for all odd n.

Circulant self-complementary graphs exist if and only every if every prime divisor of n is congruent to $1 \pmod 4$.

The only symmetric circulant sc-digraphs are the Paley graphs and Paley tournaments on a prime number of vertices.

Vertex-transitive sc-graphs exist if and only if n is the sum of two squares, while infinite vtsc-graphs exist if n is countably infinite or of order the power of the continuum. $\square$

Paley digraphs and generalisations

3.18. Let p be an odd prime, r an integer, and $\mathrm{GF}(p^r)$ be the finite field with p^r elements, with additive and multiplicative groups F_+ and F^*, respectively. It is well known that F^* is a cyclic group, say

$$F^* = \{x, x^2, x^3, \ldots, x^{p^r-1} = 1\}$$

for some generator x. Let H be the subgroup $\{x^2, x^4, \ldots, x^{p^r-1} = 1\}$ of index 2, also known as the subgroup of quadratic residues modulo p^r. Then [276] the Paley digraph $P(p^r)$ is the Cayley digraph $\mathrm{Cay}(F_+, H)$, that is,

A. $V(P(p^r)) = \{0, 1, 2, \ldots, p^r - 1\}$.

B. $E(P(p^r)) = \{(a, b) : b - a = x^{2s}$ for some s.

The Paley digraph has no loops because the unit element of F_+ is 0, which is not in H. The fact that $1 \in H$ generates F_+ tells us that $P(p^r)$ is connected.

Moreover, since $-1 = x^{\frac{p^r-1}{2}}$ is a quadratic residue if and only if $p^r \equiv 1 \pmod 4$, the Paley digraph is a graph whenever $p^r \equiv 1 \pmod 4$, and a tournament whenever $p^r \equiv 3 \pmod 4$.

The directed 3-circuit is the smallest Paley tournament, while the pentagon is the smallest Paley graph.

3.19. Theorem. *The Paley digraph $P(p^r)$ is self-complementary and symmetric.*

Proof: Since $P(p^r)$ is a Cayley graph it is vertex-transitive.

Multiplying the elements of F by $x^{2s} \in H$, or adding any element $g \in F$ are both automorphisms of $P(p^r)$. Now if $(g, g+x^{2s})$ and $(h, h+x^{2t})$ are any two arcs of the Paley digraphs, we can add $-g$, multiply by x^{2t-2s} and adding h to map the first arc onto the second. So $P(p^r)$ is arc-transitive. (In fact, the Paley tournaments are the only arc-transitive tournaments [37, 217]).

The mapping $\theta : x^r \mapsto x^{r+1}$ is a group automorphism of F_+, and so maps $P(p^r) = \mathrm{Cay}(F_+, H)$ onto $\mathrm{Cay}(F_+, F^* - H) = \overline{P(p^r)}$. $\square$

3.20. Carlitz [62] found the automorphism group of Paley graphs, while Goldberg [158] and Berggren [37] characterised the automorphism group of Paley tournaments.

Theorem. *The Paley digraphs' automorphism group is*

$$\begin{aligned}\mathcal{A}(P(p^r)) \quad = \quad & \{\phi : a \mapsto x^2\alpha(a) + y, \ \textit{where } \alpha \textit{ is a field automorphism of} \\ & GF(p^r), x, y \in \ GF(p^r), x \neq 0\}.\square\end{aligned}$$

3.21. Alspach [21] showed that when $r = 1$, we can just take α to be the identity map. In fact, he found the automorphism group of all non-trivial (i.e. non-null and non-complete) vertex-transitive digraphs on a prime number p of vertices:

$$\mathcal{A}(\mathrm{Cay}(Z_p, S)) = \{\phi : a \mapsto xa + c | x \in H(S), c \in Z_p\}.$$

where $H(S) \leq Z_p^*$ is the largest subgroup of Z_p^* such that S is a union of cosets of $H(S)$; if $\mathrm{Cay}(Z_p, S)$ is a graph, then we also stipulate that $H(S)$ must have even order. Finally, Alspach showed that the [graph] $\mathrm{Cay}(Z_p, S)$ is symmetric if and only if $S = H(S)$, that is, S is a coset of some [even order] subgroup of Z_p^*.

3.22. Chao and Wells [67] showed that the automorphism group of a non-trivial vertex-transitive digraph on a prime number p of vertices is either a cyclic group of order p, or a Frobenius group. Those with a cyclic automorphism group are called strongly vertex-transitive digraphs, which should not be confused with the concept of strongly symmetric digraphs.

So, for some α, k, $\mathcal{A}(\mathrm{Cay}(Z_p, S))$ can be expressed as $\Gamma_{\alpha,k} := \langle R, \sigma \rangle$ with the defining relations

$$R^p = 1,\ \sigma^\alpha = 1,\ \sigma^{-1} R \sigma = R^k,\ \text{where } \alpha | p-1 \text{ and } k^\alpha \equiv 1 \pmod{p}.$$

When $\alpha = 1$ we get the cyclic groups, when $\alpha > 1$ the Frobenius groups. Further, the (out)degree of the (di)graph, that is $|S|$, is a multiple of α; when $\mathrm{Cay}(Z_p, S)$ is symmetric, the degree is exactly α. The groups with odd α correspond to tournaments, while those with even α correspond to graphs.

3.23. The Paley digraphs on a prime number of vertices have a number of further special properties. First of all, the group F_+ is then a cyclic group, so that $P(p)$ is circulant. In fact there are no other circulant symmetric sc-digraphs at all [Zhang 1996a] and, on p vertices, no other symmetric sc-digraphs [Chao and Wells 1973]. Chao and Wells noted that every vertex-transitive digraph on p vertices is a union of directed Hamiltonian circuits. This brings us to an alternative definition of $P(p)$, in a short note by Nordhaus [273] which we quote verbatim:

> Let p denote a prime of the form $4n + 1$ and define row i to have the p elements $(1, 1 + i, 1 + 2i, \ldots, 1 + (p - 1)i)$, each taken modulo p, where $i = 1, 2, \ldots, r$ and $r = 2n$. Any n of these rows provide n edges disjoint hamiltonian circuits for a clock graph with p vertices labelled $1, 2, \ldots, p$ when successive elements in a row cyclically define end-points of an edge. The remaining n rows define the complementary graph, and these graphs are isomorphic when the values of i for the n rows selected are quadratic residues of p. Such graphs are examples of graphs of constant link.

By "constant link" he probably meant that the neighbourhoods of any two vertices induce isomorphic subgraphs (see [50]); this obviously follows from vertex-transitivity.

3.24. Paley digraphs are known to have a number of other properties:

A. When $p \equiv 3 \pmod 4$ and $p > k^2 2^{2k-2}$, for any k vertices of the Paley tournament $P(p)$ there is another vertex which dominates them all. [Graham and Spencer 1971]

B. When $p \equiv 1 \pmod 4$ and $p > k^2 2^{4k}$, for any $2k$ vertices of the Paley graph $P(p)$ there is another vertex adjacent to the first k but not to the last k. [Blass, Exoo and Harary 1981]

C. For any prime p and any integer r, Menger's theorems can be used to show that $P(p^r)$ has vertex-connectivity $\frac{p^r-1}{2}$. [Rao 1979a]

D. The Shannon zero-error capacity $C(G)$ of a vertex-transitive sc-graph G on n vertices is exactly $\sqrt{n}$. In particular $C(P(p^r) = p^{r/2}$. [Lovász 1979]

E. By arc-transitivity, any $n-2$ sub-tournaments of a Paley tournament are isomorphic, whereas any $n-2$ subgraphs of a Paley graph can be one of two types.

The lower bounds of A and B are by no means the best possible — see 4.33 for a discussion and application. Rao conjectured that in fact there is a vertex-transitive sc-graph of order $4k+1$ and vertex-connectivity $2k$ for any k.

3.25. Paley graphs have also been used in a topological context. White [382] defined what he called *strongly symmetrical maps*; roughly speaking, these are self-dual embeddings (in some surface) of self-complementary graphs, in which the automorphisms of the map and graph commute with the graph antimorphisms. White showed that sc-graphs with strongly symmetrical maps must be edge-transitive, and thus strongly symmetric by 3.11. He also showed that strongly symmetrical maps on n vertices exist if and only if $n \equiv 1 \pmod 8$, using the Paley graphs to establish sufficiency. We refer to White's paper for a more detailed discussion and several other results.

3.26. Paley digraphs on a prime number of vertices have been used to find lower bounds on the diagonal Ramsey numbers — specifically $R(4,4) > 17$ [163]; $R(6,6) > 101$ (which was already known), $R(7,7) > 109$, $R(8,8) > 281$ and $R(9,9) > 373$ [56]. Clapham [87] generalised the Paley graphs to establish the known bound $R(5,5) > 41$, and the new bound $R(7,7) > 113$.

Like the Paley graphs, Clapham's graphs are Cayley graphs $\mathrm{Cay}(F_+, S)$ of the additive group of the field F with $p = 4k+1$ elements. As usual, F^* is the (cyclic) multiplicative group of F generated, say, by x. We take any integer r which divides k, let H be the subgroup generated by x^2r, and define S to be the set $H \cup xH \cup \ldots \cup x^{r-1}H$. That is,

$$S := \{1, x, \ldots, x^{r-1}, x^{2r}, x^{2r+1}, \ldots, x^{3r-1}, x^{4r}, x^{4r+1}, \ldots, x^{5r-1}, \ldots\}.$$

Cay(F_+, S) is circulant by virtue of being a Cayley digraph of a cyclic group; it is loopless because $0 \notin S$, connected because $1 = x^{4k} \in S$, and it is a graph because $-1 = x^{2k} \in S$. It is also self-complementary because $\sigma : i \mapsto x^r i$ is an antimorphism (with 0 as fixed point and all other cycles having length $\frac{4k}{r}$). For $r = 1$ we get the Paley graphs. For $r \neq 1$ we get graphs that are vertex-transitive but, by Zhang's result [408], are not arc-transitive.

Evidently we can extend the definition to primes congruent to $3 \pmod 4$ (which gives us tournaments) and to any odd prime power (which gives us graphs or tournaments that are not circulant).

3.27. Guldan and Tomasta [166] used Clapham's construction to show that $R(10, 10) > 457$ and $R(11, 11) > 521$, but improved this to $R(11, 11) > 541$ by an even more general construction. Define $w := (w_1, w_2, \ldots, w_{r-1})$ where w_k is either 0 or r. Then

$$S := H \cup x^{1+w_1}H \cup x^{2+w_2}H \cup \ldots \cup x^{r-1+w_{r-1}}H.$$

As usual, 0 is not in S, but 1 and -1 are so Cay(F_+, S) is a connected circulant (but, in general, not arc-transitive) graph. When $w = (0, \ldots, 0)$ we get Clapham's graphs, and the antimorphism $\sigma : i \mapsto x^r i$ works in the general case too. As above, the definition can be extended to any odd prime power.

3.28. Kocay [224] used the Galois field with p^r elements to construct a family of vertex-transitive self-complementary 3-uniform hypergraphs. He also constructed strongly symmetric self-complementary 3-uniform hypergraphs G^* which subsume the Paley graphs. That is to say, for any vertex v, if $E_v := \{\{v, w, z\} \in E(G^*\}$ is the set of hyperedges containing v, then the edges $\{wz | \{v, w, z\} \in E_v\}$ define a graph that is isomorphic to $P(p^r)$.

Characterisation of some vertex-transitive sc-digraphs

3.29. Vertex-transitive digraphs, as mentioned above, are just coset digraphs. There is, so far, no characterisation of the vertex-transitive self-complementary digraphs, but there are two special cases where we can obtain a suitable algebraic characterisation. Zhang [409] gave the following characterisation of symmetric sc-digraphs.

Theorem. *A digraph is symmetric and self-complementary if and only if it is isomorphic to a Cayley digraph* $Cay(V_+, O_H)$ *where*

A. V_+ *is the additive group of the vector space* V *of dimension* r *over the finite field with* p *elements, where* p *is an odd prime.*

B. O_H *is an orbit of a group* H, $H \subset \overline{H} \subset GL(V)$, $[\overline{H} : H] = 2$, $\overline{H}$ *is transitive on* $V - \{0\}$ *but* H *is not transitive on* $V - \{0\}$. □

As noted in 3.6, when $p^r \equiv 1 \pmod 4$ we get the symmetric sc-graphs and, when $p^r \equiv 3 \pmod 4$, the Paley tournament. Moreover, if we also require that the sc-digraphs be circulant, we are left with only the Paley graphs and tournaments.

3.30. The second case where we have complete information is when the vertex-transitive sc-digraph has a prime number p of vertices. Turner showed that any vertex-transitive digraph on p vertices must be circulant [373]. He also gave the following result:

Theorem. *Two circulant graphs* $\mathrm{Cay}(Z_p, H)$ *and* $\mathrm{Cay}(Z_p, H')$ *on a prime number* p *of vertices are isomorphic if and only if* $H' = aH$ *for some integer* a *co-prime to* p.

This has been generalised to prime order digraphs [67, 111], and also [261, 262] to digraphs on $n = \epsilon p_1 p_2 \cdots p_s$ vertices, where the p_i's are pairwise distinct odd primes and $\epsilon \in \{1, 2, 4\}$; it is false for all other $n \geq 18$ [277].

So, for all these values of n, a circulant digraph $\mathrm{Cay}(Z_n, H)$ is self-complementary if and only if there exists an integer a co-prime to n such that

$$aH = Z_n - \{0\} - H.$$

The prime order case was first pointed out by Ruiz [333] and then treated in more detail by Chia and Lim [78]. We first note that the multiplicative group Z_p^* is cyclic, and any subgroup A of even order 2ω is also cyclic and will thus have a unique subgroup A_1 of order ω; we let $A_2 := A - A_1$. Every coset xA can be split similarly into xA_1 and xA_2.

3.31. Theorem[Chia and Lim 1986]**.** *Let $D = Cay(Z_p, S)$ be a vertex-transitive digraph on an odd prime number p of vertices. Then D is self-complementary if and only if, for each even order subgroup $A \leq Z_p^*$, and for each coset xA,*

$$\text{either } H \cap xA = xA_1 \text{ or } H \cap xA = xA_2.$$

Strongly regular sc-graphs

3.32. So far we have only discussed strongly regular sc-graphs (srsc-graphs) to find out where they fit in the first hierarchy. We now take a look at some of their properties, starting with their parameters.

Proposition[Seidel 1976]**.** *A strongly regular sc-graph G has parameters $(4k+1, 2k, k-1, k)$, for some k; and eigenvalues $2k$, $\frac{-1-\sqrt{4k+1}}{2}$, $\frac{-1+\sqrt{4k+1}}{2}$, with multiplicities 1, $2k$, $2k$, respectively.*

Proof: By 3.2.A, G must have $4k+1$ vertices and degree $2k$. Now take a vertex $v \in F(G)$, and consider some antimorphism σ which fixes v. We define A to be the set of neighbours of v, and $B = V(G) - A - v$. Notice that $\sigma(A) = B$, $\sigma(B) = A$ and $|A| = |B| = 2k$. Since G is strongly regular, $\mu \cdot 2k = |E[A, B]| = 2k^2$, and therefore $\mu = k$. Finally, if v and w are any two adjacent vertices, with λ common neighbours and degree $2k$, then in $\overline{G}$, v and w will be non-adjacent and have k common neighbours, λ common non-neighbours and degree $2k$. Thus $\lambda = k-1$.

The eigenvalues and multiplicities of a strongly regular graph can be derived directly from the parameters (n, ρ, λ, μ) using standard formulas for which we refer to Biggs [42, p. 20]. $\square$

Curiously [26], every strongly regular Cayley graph (even those which are not self-complementary) based on an abelian group and with $\lambda - \mu = -1$ must also have parameters $(4k+1, 2k, k-1, k)$, with the exception of some graphs with parameters $(243, 22, 1, 2)$ and their complements.

For srsc-graphs, we can say a bit more.

3.33. Lemma[Kotzig[2]]. *For a regular sc-graph on $4k+1$ vertices, the following conditions are equivalent*

A. G is strongly regular,

B. each edge is on exactly $k-1$ triangles,

C. each pair of non-adjacent vertices has exactly k common neighbours.

Proof: That A implies B and C follows from Proposition 3.32, while B $\Rightarrow$ A follows from 3.8, and it is easy to check that C implies B. $\square$

3.34. Seidel [351] showed that srsc-graphs on n vertices can only exist when n is a sum of two squares; and that their adjacency matrix A must satisfy

$$A^2 = k(J+I) - A, \ AJ = 2kJ.$$

Mathon [247] remarked that these conditions are necessary but not sufficient; however, he did not give any particular examples.

The fact that srsc-graphs do exist when n is a prime power was known as early as 1967 [157, 351]; it also follows from the existence of symmetric sc-graphs, which we know are all strongly regular.

Up to $n = 29$ the only strongly regular sc-graphs are the Paley graphs. Non-Paley examples are difficult to find, but a few were found for $n = 37$, 41 and 49 [58]; see also [211].

Liu [239] also tackled these issues, but his results — srsc-graphs exist for certain primes, and the srsc-graph of order 13 is unique — do not cover any new ground.

3.35. Rosenberg [331] made a detailed study of self-complementary graphs using boolean techniques. He obtained a description of the sc-graphs on $4k+1$ vertices in terms of certain $0-1$ parameters, and found the constraints these

[2] Communicated to Rosenberg [331]

parameters must satisfy for the graph to be regular or strongly regular. The systematic generation of srsc-graphs is thus reduced to the solution of certain integer equations. However, this is nowhere near as convenient as having an explicit counting formula, something which still remains elusive.

Mathon made a similar attempt in [247] to count the strongly regular self-complementary graphs on up to 49 vertices. He first tried to restrict the search for srsc-graphs as far as possible, and then generated them systematically, providing a wealth of information about their structure, such as the sizes of their orbits, cycle structures of their antimorphisms, and number of K_4's, K_5's, K_6's and K_7's.

We give the numbers of non-isomorphic srsc-graphs in Table 3.1. As noted previously, there is only one graph for each feasible order up to $n = 29$, namely, the Paley graph of order n. The next Paley graph is on 49 vertices, so all the others are non-Paley. We note that many of these graphs, especially on 45 vertices, have very small automorphism groups (sometimes just of size 2), while others, especially the Paley graphs, have hundreds of automorphisms.

n	5	9	13	17	25	29	37	41	45	49
$\overline{sr}_n$	1	1	1	1	1	1	2	4	22	5

Table 3.1: Strongly regular sc-graphs on n vertices

Mathon also made a list of open problems:

A. Can a srsc-graph on $n = 8k + 1$ vertices have an antimorphism of order 4. (Up to $n = 49$, none do).

B. Can a srsc-graph on n vertices, n a prime power, have an antimorphism whose non-trivial cycles have equal length. The only examples up to 49 vertices are for $n = 45$, which is not a prime power.

C. Do there exist srsc-graphs on $n = m^2$ vertices which belong to a switching class of a Steiner system? All known examples are of Latin type.

D. Find infinite families of srsc-graphs on non-Paley type. In particular, are there any srsc-graphs at all for $n = 65$ and $n = 85$? [Mathon 1978]

3.36. One of the restrictions which Mathon found has to do with the cycle structures of the antimorphisms. Recall that every antimorphism (with $n =$

$4k+1$ vertices) is the antimorphism of some regular sc-graph. If we add the condition of strong regularity, then this is no longer true, and not only because n must be the sum of two squares.

For every σ of order $2^s(2t+1)$, $\tau := \sigma^{2t+1}$ is an antimorphism of order 2^s with one cycle of length 1, the rest having lengths $2^{l_1}, \ldots, 2^{l_\omega}$, where $l_1 \geq \cdots \geq l_\omega \geq 2$. We call τ a *basic antimorphism*, and represent its cycle lengths by the vector $\mathbf{l}(\tau) = (l_1, \ldots, l_\omega)$.

Theorem[Mathon 1988]**.** *Let G be a srsc-graph on $4k+1$ vertices with a basic complementing permutation τ, where $\mathbf{l}(\tau) = (l_1, \ldots, l_\omega)$. If $l_1 = \cdots = l_\gamma = c > l_{\gamma+1}$ for some $1 \leq \gamma < \omega$, then*

$$k \leq 2^{c-1}\omega - 1.$$

If, moreover, two of the exponents l_α differ by more than 1, that is $c - l_\delta \geq 2$ for some $\delta > \gamma$, then

$$k \leq \min_{\gamma \leq t < \omega} \{\rho_t - \lceil \rho_t(\upsilon_t - 1)/(2\upsilon_t - 1) \rceil\},$$

where

$$\upsilon_t = 2^{c-l_t-1} \text{ and } \rho_t = 2^{c-1}\gamma + \sum_{i+1}^{t-1} 2^{l_i-1}.\square$$

Kotzig's problems

3.37. We finally come to Kotzig's problems [227, Problems 2–7]. These were originally stated as follows:

For a self-complementary graph G define the three sets

$$\begin{aligned} F(G) &:= \{u \in V(G) | \exists \sigma \in \overline{\mathcal{A}}(G), \sigma(u) = u\}, \\ \hat{F}(G) &:= \{u \in V(G) | t(u) = k(k-1)\} \text{ (where } G \text{ is regular), and} \\ N(G) &:= \{uv \in E(G) | \exists \sigma \in \overline{\mathcal{A}}(G), \sigma(u) = v\}. \end{aligned}$$

A. Is it true that every regular self-complementary graph has an antimorphism where all cycles are of length 4, except for a single fixed vertex?

B. Characterise the set $F(G)$ of a regular sc-graph G. Is it true that $F(G) = \hat{F}(G)$ for a regular sc-graph G?

C. Characterise the set $N(G)$ for a sc-graph G.

D. Is is true that a regular sc-graph G is strongly regular iff $F(G) = V(G)$ and $N(G) = E(G)$?

E. Is it true that a strongly regular sc-graph on $n = 4k + 1$ vertices exists whenever n is a sum of two squares? What is the smallest integer k for which there are at least two non-isomorphic srsc-graphs with $4k+1$ vertices?

Before stating the last problem we need some terminology. Let $\overline{R}_k$ denote the set of all regular sc-graphs with $4k+1$ vertices, let $\mu(G)$ be the maximal number of mutually edge-disjoint Hamiltonian cycles of G, and define

$$\mu(k) = \min_{G \in \overline{R}_k} \{\mu(G)\}.$$

Clearly $\mu(k) \leq k$, and it has been proved [212] that $\mu(k) \geq \left\lfloor \frac{k}{3} \right\rfloor$. Kotzig stated that

- a construction of rsc-graphs with $\mu(G) = k$ was known for every natural number k (see 3.23 for the case where k is prime),
- $\mu(1) = 1$ and $\mu(2) = 2$, but $\mu(3)$ was unknown, and
- there was no known example of a graph $G \in \overline{R}_k$ with $\mu(G) < k$.

His last problem was:

F. Find exact values of $\mu(k)$ for small $k > 2$.

3.38. We have seen (3.10) that the definition of $\hat{F}(G)$ can be extended to cover any graph G, even if it is not regular or self-complementary:

$$\hat{F}(G) = \{u \in V(G) | t(u) = \overline{t}(u)\}.$$

With this definition it is evident that

$$F(\overline{G}) = F(G) \subseteq \hat{F}(G) = \hat{F}(\overline{G}).$$

By the time Kotzig posed his problems, $F(G)$ had already been characterised for graphs by Robinson [321] as the unique odd orbit of G; he later showed [324] that for any self-complementary digraph, $F(G)$ is the unique orbit fixed by any and every antimorphism. Rao proved as much, and more, in [306], showing how every antimorphism induces an involution on the orbits of G and thus characterising $N(G)$ (see 1.34). When combined with Robinson's second result, this can be extended to sc-digraphs (see 5.7), although $F(G)$ need not be the unique odd orbit in the general case.

The first part of problem E is listed as an open question in 3.35, while the answer to the second part (see Table 3.1) is $k = 9$. We do not know of any progress on problem F.

Using Nair, Vijayakumar and Rao's results, and calling the one cycle (if any) of an antimorphism the *trivial* cycle, we can restate the remaining questions as follows:

A. Is it true that every regular self-complementary graph has an antimorphism where all non-trivial cycles are of length 4?

B. Is it true that $F(G) = \hat{F}(G)$ for every regular sc-graph G?

D. Is is true that a regular sc-graph G is strongly regular iff it is vertex-transitive?

It turns out that the answer to all three is 'No', but the construction of counter-examples is not an easy matter.

3.39. We will tackle these problems in reverse order, the last one being easiest. The first counterexample to problem D was given by Ruiz [336], who noted that there are exactly two vertex-transitive self-complementary graphs with 13 vertices — the Paley graph, and the circulant graph $\mathrm{Cay}(Z_{13}, \{1, 3, 6, 7, 10, 12\})$. The latter, shown in Figure 3.2 is a counterexample to Kotzig's problem.

We saw in 3.16 how Rao [306] used the composition of Paley graphs to construct vertex-transitive sc-graphs. He used the same method to give an infinite family of counterexamples to problem D. First he characterised the connected strongly regular compositions:

Lemma. *The graph $G(H)$ is connected and strongly regular if and only if*

A. G and H are both complete graphs, or

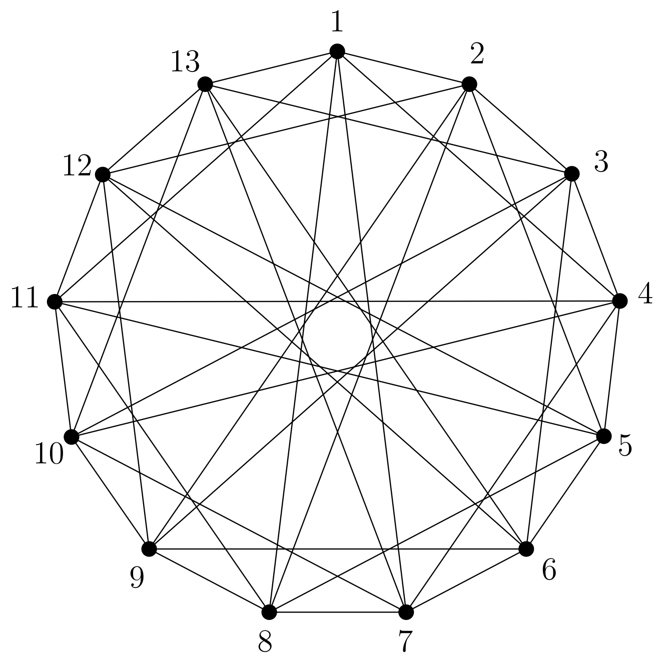


Figure 3.2: $\mathrm{Cay}(Z_{13}, \{1, 3, 6, 7, 10, 12\})$

B. G is complete and H is complete multipartite, or

C. G is complete and H is a null graph, or

D. G is complete multipartite and H is a null graph. □

So the composition of two or more vertex-transitive sc-graphs is a counterexample to problem D. Using Paley graphs, we can construct such counterexamples for all n which can be expressed as $p_1^{r_1} \dots p_s^{r_s}$, where $p_1, \dots, p_s$, is a list of at least two primes (not necessarily distinct) and $p_i^{r_i} \equiv 1 \pmod 4$ for all i. Equivalently, n is a sum of two squares but not a prime power.

Both vertex-transitive and strongly regular sc-graphs exist only if n is the sum of two squares. For vertex-transitive sc-graphs the converse is true 3.16, but for strongly-regular sc-graphs the question is still open. If there are any n of the form $a^2 + b^2$ for which no srsc-graphs exist, the vtsc-graphs on n vertices would automatically be counterexample s.

We could also ask for strongly regular sc-graphs which are not vertex-transitive, but so far no examples have been found.

3.40. Rao [306] also claimed to have constructed an infinite family of counterexamples to question B, one for each feasible order $n \geq 9$. But Nair and Vijayakumar [263, 265] pointed out that the construction only works when $n = 9$; for the sake of this discussion we will denote Rao's graph by G_9 (Figure 3.3). They constructed an infinite family of counterexamples, although ironically their original proof was faulty.

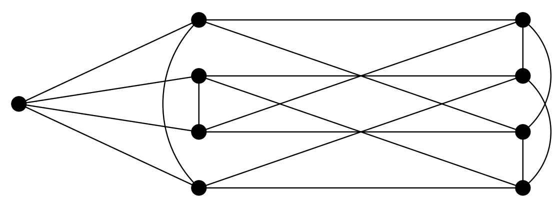


Figure 3.3: G_9

Since $F(G) = V(G)$ if and only if G is vertex-transitive; and, when G is regular, $\hat{F}(G) = V(G)$ if and only if G is strongly vertex triangle regular (SVTR), we have something to go by when looking for counterexamples — any sc-graph that is strongly vertex triangle regular, but not vertex-transitive, will do. It turns out that G_9 is such a graph, that is $|\hat{F}(G_9)| = 9 > |F(G_9)|$; in fact, $|F(G_9| = 1$.

Nair and Vijayakumar constructed two other graphs satisfying this criterion, G_{17} on 17 vertices and G_{33} on 33. Essentially they took a copy of a circulant graph on 8 or 16 vertices, and a copy of its complement, joined them together in a particular way, and then added another vertex. They suggested that it might be possible to extended this method to any larger $n = 2^k + 1$.

One would like to be able to compose these graphs (that is, G_9, G_{17} and G_{33}) to obtain larger and larger counterexamples, but it is not clear that this can be done. Nair and Vijayakumar did prove that if G and H are any two graphs, and $G(H)$ their composition (obtained by replacing every vertex of G by a copy of H, and every edge by a corresponding bundle of all possible edges, as in refintrod-26), then

A. if G and H are both SVTR, then so is $G(H)$.

It can also easily be checked (c.f. 1.26) that

B. if G and H are both self-complementary, then so is $G(H)$.

The stumbling block is vertex-transitivity. Obviously

C. if G and H are vertex-transitive, then so is $G(H)$.

Nair and Vijayakumar apparently assumed a partial converse — that if neither G nor H is vertex-transitive, then $G(H)$ is not — so that composing any number of G_9's, G_{17}'s and G_{33}'s in any order would give us counterexamples. While it seems plausible, they offered no proof. However, after some correspondence with the author this was proved for certain cases.

3.41. Theorem. *Let G and H be regular self-complementary graphs. Let $N(u)$, $N(x)$ and $N(u,x)$ be the neighbourhoods of $u \in V(G), x \in V(H)$ and $(u,x) \in G(H)$. Then*

A. *If for some $u, v \in V(G)$, the degree sequences of $N(u)$ and $N(v)$ are different, then $N(u,x)$ and $N(v,x)$ will also have different degree sequences.*

B. *If for some $x, y \in V(H)$, the degree sequences of $N(x)$ and $N(y)$ are different, then $N(u,x)$ and $N(u,y)$ will also have different degree sequences.*

C. *If for some $x, y \in V(H)$, $N(x) \ncong N(y)$, then $N(u,x) \ncong N(u,y)$.*

Proof: Let G, H have degrees d_G, d_H, and orders n_G, n_H respectively.

The neighbourhood $N(u,x)$ of (u,x) consists of a copy of $N(x)$, with every vertex joined to all the vertices of a copy of $N(u)(H)$. There are d_H vertices in $N(x)$, and d_G copies of H in $N(u)(H)$.

Thus a vertex (u,z) in N(u,x), where $z \in N(x)$, is adjacent to

- $d_{N(x)}(z)$ vertices in the copy of $N(x)$, and
- n_H vertices in each of the d_G copies of H.

So denoting the degree of (u,z) in $N(u,x)$ by $d'(u,z)$, we have

$$d'(u,z) = d_{N(x)}(z) + d_G n_H.$$

A vertex (w,h) in N(u,x), where $w \in N(u), h \in V(H)$, is adjacent to

- the d_H vertices of the copy of $N(x)$,

- d_H vertices of the form (w, h'), where $h \sim h'$ in H, and
- n_H vertices in each of $d_{N(u)}(w)$ copies of H.

The total degree of (w, h) in $N(u, x)$ is thus

$$d'(w, h) = 2d_H + d_{N(u)}(w) \cdot n_H.$$

Parts A and B can now be seen to be true. To establish the last part we note that $2d_H = n_H - 1$, so that

$$d'(w, h) < [d_{N(u)}(w) + 1] \cdot n_H \leq d_G n_H < d'(u, z).$$

Similar considerations hold for the degree of any vertex (u, z') in $N(u, y)$. Thus, any isomorphism of $N(u, x)$ onto $N(v, x)$ must induce an isomorphism of $N(x)$ onto $N(y)$ to preserve the degrees. But this is impossible if $N(x) \not\cong N(y)$. □

3.42. Now G_9, G_{17} and G_{33} all contain vertices whose neighbourhoods are non-isomorphic, or even have different degree sequences, so we can compose them with each other, and with any other SVTRSC graph to obtain counterexamples to Kotzig's conjecture. Using previous results on the existence of circulant sc-graphs we can thus state:

Theorem. *There are strongly vertex triangle regular self-complementary graphs which are not vertex-transitive, of order n, for all $n = 9^\alpha 17^\beta 33^\gamma N$, where at least one of α, β, γ is not zero, and N is a sum of two squares.* □

We note as an aside that Ruiz [338] showed that, for all $k > 1$, there are graphs on $4k + 1$ vertices which are regular and self-complementary but not vertex-transitive. His example was the C_5-join of $(\overline{K}_k, K_k, K_1, K_k, \overline{K}_k)$. In fact $|F(G)| = 1$ for these graphs, since the central K_1 is the only vertex which is in just two cliques. However, they will not do as counterexamples to Kotzig's problem B since $|\hat{F}(G)| = 1$ as well.

Antimorphisms with unequal cycle lengths

3.43. Hartsfield [198] gave a single counterexample to question A. (Xu [393] later produced a counterexample too). Hartsfield's graph H (see Figure 3.4) has 13 vertices and four antimorphisms, which all have a 1-cycle, a 4-cycle and an 8-cycle:

$$\begin{aligned}
\sigma &= (1)(2\,3\,4\,5)(6\,7\,8\,9\,10\,11\,12\,13)\\
\sigma^3 &= (1)(2\,5\,4\,3)(6\,9\,12\,7\,10\,13\,8\,11)\\
\sigma^5 &= (1)(2\,3\,4\,5)(6\,11\,8\,13\,10\,7\,12\,9)\\
\sigma^7 &= (1)(2\,5\,4\,3)(6\,13\,12\,11\,10\,9\,8\,7).
\end{aligned}$$

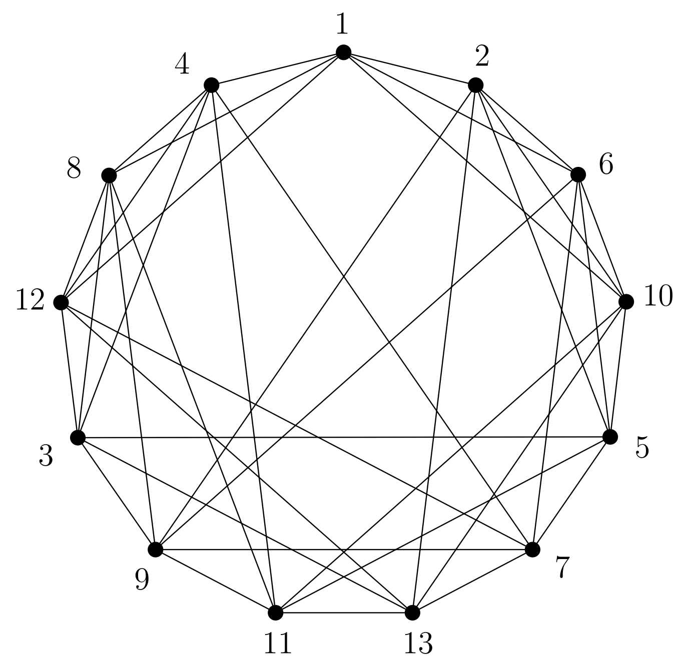


Figure 3.4: Hartsfield's graph

Its automorphism group, in fact, is the following:

$$\begin{aligned}
\sigma^2 &= (1)(2\,4)(3\,5)(6\,8\,10\,12)(7\,9\,11\,13)\\
\sigma^4 &= (1)(2)(3)(4)(5)(6\,10)(7\,11)(8\,12)(9\,13)\\
\sigma^6 &= (1)(2\,4)(3\,5)(6\,8\,10\,12)(7\,9\,11\,13)\\
\sigma^8 &= \textsc{id}.
\end{aligned}$$

We note that the vertex 1 is the only fixed vertex of H, and its neighbourhood is $N = \{2, 4, 6, 8, 10, 12\}$. We define $\overline{N} := \{3, 5, 7, 9, 11, 13\}$.

3.44. What if we broaden the scope of Kotzig's problem to "Is it true that every self-complementary graph on n vertices has an antimorphism with non-trivial cycles of equal length?". This is trivially true for $n \leq 9$, but Hartsfield's graph shows that it is not true when $n = 13$.

If we remove the vertex 1 we get a counterexample of order 12. To see this, note that any antimorphism of $H - \{1\}$ must exchange the vertices of N and $\overline{N}$ because they have distinct degrees. So the antimorphisms of $H - \{1\}$ can be extended to an antimorphism of H and, thus, must have cycles of length 4 and 8.

Now, if J is any counterexample of order n, we can add a P_4 to give a sc-graph with end-vertices as in 1.13. Any antimorphism of this new graph J' with $n + 4$ vertices must map the P_4 to itself and J onto itself; so it too will contain cycles of unequal lengths. In this manner we obtain an infinite family of counterexamples for each feasible n:

Proposition. *There is a sc-graph on n vertices whose antimorphisms all have non-trivial cycles of unequal length (specifically, at least one cycle of length 4 and at least one of length 8) if and only if $n \equiv 0$ or $1 \pmod 4$, $n \geq 12$.* □

3.45. Apart from Hartsfield's graph, the graphs we constructed are not regular, so they are not counterexamples to Kotzig's problem. With a little work, however, we can use Hartsfield's graph (or any other counterexample, if one is found) as the basis for an infinite family of counterexamples on $n = 13 + 8k$ vertices, as follows. Our construction is loosely based on one by Colbourn and Colbourn [94].

A *clique* is a complete subgraph which is not included in a larger complete subgraph. A maximal independent set is just the complement of a clique — an independent set not included in any larger independent set; for convenience we abbreviate this to *maxiset*. It can be checked that a largest clique of H has size 4, and thus a largest maxiset of H also has size 4.

For any natural number k, we construct $H(k)$ as follows (see Figure 3.5 for a sketch). We add two K_{2k}'s and two $\overline{K}_{2k}$'s. In each of these four graphs, the first k vertices are joined to every vertex of N, the last k vertices are joined to every vertex of $\overline{N}$. Every vertex of the K_{2k}'s is also joined to the vertex 1. Finally, every vertex of each $\overline{K}_{2k}$ is joined to every vertex of the other $\overline{K}_{2k}$, and every vertex of one of the K_{2k}'s.

This graph is regular of degree $4k+6$ and self-complementary. For $k \geq 3$,

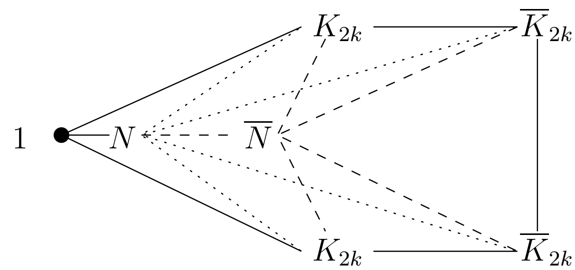


Figure 3.5: $H(k)$

the following facts can also be checked (the cliques of H itself get in the way for $k = 1$ or 2):

- the vertex 1 is included in exactly two cliques of size $2k + 1$ which intersect in the vertex 1, and exactly two maxisets of size $2k + 1$
- each vertex in the K_{2k}'s which we added is contained in $2k + 1$ cliques of size $2k + 1$, and one maxiset of size $2k + 1$;
- each vertex in the $\overline{K}_{2k}$'s which we added is contained in $2k+1$ maxisets of size $2k + 1$, and one clique of size $2k + 1$;
- the vertices of N and $\overline{N}$ are not contained in any cliques or maxisets of size $2k + 1$

So any antimorphism must map H onto itself, and thus must induce one of the antimorphisms of H mentioned above. So we have a family of regular graphs $H(k)$ on $n = 13 + 8k$ vertices, $n \geq 37$, for which every antimorphism must have non-trivial cycles of length 1, 4 and 8.

We noted previously that $H - \{1\}$ is an almost regular sc-graph whose antimorphisms all have cycles of unequal length; the same reasoning is valid for $H(k) - \{1\}$. So we have the following:

Proposition. *For all integers $n = 13 + 8k$ [$n = 12 + 8k$], except possibly $n = 20, 21, 28$ or 29, there is a regular [almost-regular] sc-graph whose antimorphisms all contain non-trivial cycles of unequal length, specifically at least one cycle of length 4 and at least one of length 8.* □

Open Problems[3]

3.46.

A. For which integers n do there exist

(1) vertex-transitive sc-graphs [Zelinka 1979]. In particular [Fronček, Rosa and Širáň 1996] do these exist for $n = pq$, where graphs of order pq where p and q are distinct primes both congruent to 3 (mod 4)?

(2) strongly regular graphs [Kotzig 1979]

(3) Cayley sc-graphs.

(4) reflexible [semi-reflexible, anti-reflexible] groups.

B. Are there any circulant sc-digraphs that are neither graphs nor tournaments?

C. [Alspach 1970] Do there exist vertex-transitive sc-tournaments that are not circulant?

D. Do there exist vertex-transitive sc-graphs [digraphs, tournaments] which are not Cayley graphs?

E. Construct further counterexamples (preferably an infinite family) to Kotzig's problem 3.37.B — that is, regular sc-graphs with $F(G) \neq \hat{F}(G)$.

F. (Kotzig's last problem). Find bounds (or, for small k, exact values) for $\mu(k)$, as defined in 3.37.F.

G. Construct strongly regular sc-graphs that are not Paley graphs, and at least one that is not vertex-transitive.

H. Is it true that for every natural number k there is a vertex-transitive self-complementary graph G on $4k+1$ vertices with vertex connectivity $\kappa(G) = 2k$? [Rao 1979a]

[3] See also 3.35

Chapter 4

Self-complementarity

4.1. What importance do self-complementary graphs have for the rest of graph theory? We saw in 1.65 that to prove or disprove the Strong Perfect Graph Conjecture it is enough to consider only self-complementary graphs. We now see that the same is true for the isomorphism problem; in fact, even checking whether a graph is self-complementary is equivalent to the isomorphism problem. We will consider how to measure the degree of self-complementarity of graphs in general, and look at ways of generating the sc-graphs, which gives us the opportunity to tackle antimorphisms in further depth. Finally, we see the role that self-complementary graphs play in the reconstruction conjecture and in defining certain codes.

The isomorphism problem

4.2. One of the most basic tasks when dealing with a particular class of graphs is to distinguish members of the class from each other, and from non-members. These are the *isomorphism problem* and the *recognition problem*, respectively, for that class. For graphs in general the isomorphism problem is as yet intractable - there is no known polynomial solution, and it is not even known if one exists.

Obviously, any test for determining whether two graphs or digraphs are isomorphic will also work for sc-graphs and sc-digraphs. It can also be used to tell us whether a given graph is self-complementary, as we just need to check whether it is isomorphic to its complement.

It might be expected that the isomorphism problem would in fact be easier to tackle when restricted to self-complementary graphs or digraphs, because of their strong structural properties. It turns out, however [Colbourn and Colbourn 1978, 1979], that the isomorphism problem for sc-graphs is polynomially equivalent to the general isomorphism problem; we say that it is *isomorphism complete*. Even if we just want to know whether a graph or digraph is self-complementary, the complexity is the same. This makes it improbable that there will be any simple and quick test for recognising sc-graphs; for example, comparing the chromatic polynomial of a graph with that of its complement will not tell us whether it is self-complementary (see 1.59).

Recognition and isomorphism of self-complementary graphs therefore take on added importance. They could provide a cure for what has been nicknamed the isomorphism disease, and even settle the famous (or notorious) question of whether P is equal to NP, as we shall see.

4.3. Algorithmic problems are grouped into several classes according to their complexity. The class P contains those problems which can be solved in polynomial time. The class NP (for "Non-deterministic polynomial") contains those problems for which any proposed solution can be checked in polynomial time, so P $\subseteq$ NP. Of course, finding the solution in the first place is another matter altogether; for many problems, such as the Travelling Salesman Problem, the best known solution so far is highly inefficient — for graphs with a few hundred or thousand vertices, it could take longer to find an optimal solution than the time which has elapsed since the Big Bang! It is thought that these problems are inherently difficult, but there is as yet no proof of this, so the question "P = NP?" remains open.

The NP-complete problems are the hardest ones in NP, in the following sense. If A is an NP-complete problem, then any polynomial-time solution for A would give us a polynomial time solution for all the other problems in NP, thus showing that P = NP. In the last three decades hundreds of problems have been shown to be NP-complete, among them the Travelling Salesman Problem, deciding whether a graph G is Hamiltonian, and deciding whether G is k-colourable (for some fixed $k \geq 3$). It is ironic that the Hamiltonian problem has an $O(n^2)$ solution for self-complementary graphs (see 2.16), whereas the isomorphism problem (which, for graphs in general, is thought to be easier) has no known polynomial solution for sc-graphs.

There are polynomial time algorithms for testing the isomorphism of graphs with bounded valency [242], or graphs with no non-trivial automorphisms [29, 218]; for planar graphs there is even a linear time algorithm [209]. But we have shown in 1.21, 1.29 and 1.46 that the class of self-complementary graphs does not satisfy any of these criteria.

4.4. Graph isomorphism is in NP, since if we are given two graphs G, H, and a bijection from $V(G)$ to $V(H)$, we can check in polynomial time whether it is in fact an isomorphism. It is not known whether graph isomorphism is NP-complete, but it is known that if P $\neq$ NP then there must be a third class of problems whose complexity is strictly between P and NP, and the graph isomorphism problem is thought to be a suitable candidate for such an intermediate class.

Kobler *et al.* [223] give a detailed treatment of algorithmic issues and graph isomorphism in particular, showing several ways in which it behaves differently from most NP-complete problems. Significantly, they prove that if graph isomorphism is NP-complete then P = NP, which makes it unlikely that it is.

Our current state of knowledge therefore leaves three possibilities:

A. graph isomorphism is NP-complete and thus polynomial because P = NP; or

B. graph isomorphism is polynomial, but not NP-complete, and thus P $\neq$ NP; or

C. graph isomorphism is neither polynomial nor NP-complete, and so P $\neq$ NP.

By Kobler's and Colbourn's results, exactly the three same possibilities occur for self-complementary graphs, and we have the following:

Theorem. *P = NP if and only if the recognition or isomorphism of sc-graphs is NP-complete.*

Proof: If P = NP then the recognition or isomorphism of sc-graphs must be polynomial, and thus NP-complete. Conversely, if they are NP-complete, then graph isomorphism must also be NP-complete and thus, by Kobler *et al.*, we have P = NP. □

Dor and Tarsi [108] settled a conjecture by Holyer, to the effect that, for a given connected graph H with three edges or more, determining whether a graph G has a decomposition into factors isomorphic to H is NP-complete. The problem of self-complementarity is somewhat related; in this case, the graph H can vary, but must always have $n(n-1)/4$ edges, and G is always K_n, where $n = |V(H)|$. Whether Dor and Tarsi's results will be of any help remains to be seen.

We now turn to the proof of Colbourn and Colbourn's results.

4.5. Theorem. *The digraph isomorphism problem is polynomial if and only if the self-complementary digraph isomorphism problem is polynomial.*
Proof: Let us say we want to check whether two digraphs D_1, D_2, on n vertices are isomorphic. We form S_{D_1,D_2} by substituting D_1 and $\overline{D_2}$ for the vertices of $\vec{P_2}$ (see Figure 4.1). Obviously, if D_1 and D_2 are isomorphic, then

$$D_1 \longrightarrow \overline{D_2} \qquad\qquad \overline{D_1} \longleftarrow D_2$$

Figure 4.1: S_{D_1,D_2} and $\overline{S_{D_1,D_2}}$

S_{D_1,D_2} is self-complementary. Conversely, let S_{D_1,D_2} be self-complementary. Since every vertex in the copy of D_1 has outdegree at least n, and every vertex in the copy of $\overline{D_2}$ has outdegree at most $n-1$, any antimorphism of S_{D_1,D_2} will map D_1 to D_2. So

$$D_1 \cong D_2 \Leftrightarrow S_{D_1,D_2} \text{ is self-complementary.}$$

If instead of $\overline{D_2}$ we use the converse, D_2', we see that

$$D_1 \cong D_2 \Leftrightarrow S_{D_1,D_2} \text{ is self-converse.}$$

Moreover, if we can check for self-complementarity or self-converseness in $O(n^r)$ time, for some constant r, then we can check any two digraphs for isomorphism in $O(2^r n^r + n^2) = O(n^r)$ time[1]. $\square$

4.6. Theorem. *The digraph isomorphism problem is polynomial if and only if the recognition of self-complementary digraphs is polynomial.*

[1] We must have $r \geq 2$, because any isomorphism algorithm must check all $n(n-1)$ pairs of edges

Proof: Let us say we want to check whether two digraphs D_1, D_2, on n vertices are isomorphic. We form S_{D_1} by substituting D_1 and $\overline{D_1}$ for the vertices of $\vec{P_2}$; and S_{D_2} by substituting D_2 and $\overline{D_2}$ (see Figure 4.2). S_{D_1} and

$$D_1 \longrightarrow \overline{D_1} \qquad\qquad \overline{D_2} \longleftarrow D_2$$

Figure 4.2: S_{D_1} and S_{D_2}

S_{D_2} are both self-complementary. As above, any isomorphism of S_{D_1} and S_{D_2} must map D_1 to D_2, so we have

$$D_1 \cong D_2 \Leftrightarrow S_{D_1} \cong S_{D_2}.$$

Similarly, if we use the converses, D_1' and D_2', instead of $\overline{D_1}$ and $\overline{D_2}$, we see that we can check D_1 and D_2 for isomorphism by testing the isomorphism of self-converse graphs. □

4.7. We note that if D_1 and D_2 are tournaments, then the digraphs S_{D_1,D_2}, S_{D_1} and S_{D_2} will also be tournaments, so that we have the following.

Theorem. *The tournament isomorphism problem is polynomial if and only if the recognition or isomorphism of sc-tournaments is polynomial.* □

However, while the graph and digraph isomorphism problems are known to be polynomially equivalent, it is uncertain whether they are equivalent to the tournament isomorphism problem. This can be partly explained as follows. The equivalence theorems all depend on representing an arbitrary digraph or pair of digraphs by a unique self-complementary or self-converse digraph. The automorphism group of the original digraph will be a subgroup of the automorphism group of the sc-digraph; we say that the automorphism group is preserved. But any tournament representation of a graph or digraph cannot always preserve the automorphism group — graphs and digraphs can have automorphism groups of any order, while tournaments can only have odd-order automorphism groups. See also 1.32.

4.8. Theorem. *The recognition and isomorphism problems for sc-graphs are polynomially equivalent to the isomorphism problem for graphs in general.*

Proof: We first show that testing isomorphism of two graphs G, H, on n vertices is equivalent to testing self-complementarity of a graph on $4n$ vertices. We form $\mathcal{P}(G,\overline{H})$ by substituting copies of G for the end-vertices of a P_4, and copies of $\overline{H}$ for the interior vertices (see Figure 4.3). Obviously

$$G \text{ —— } \overline{H} \text{ —— } \overline{H} \text{ —— } G \qquad\qquad H \text{ —— } \overline{G} \text{ —— } \overline{G} \text{ —— } H$$

Figure 4.3: $\mathcal{P}(G,\overline{H})$ and $\overline{\mathcal{P}(G,\overline{H})}$

if G and H are isomorphic, then the new graph will be self-complementary.

Conversely, any isomorphism between $\mathcal{P}(G,\overline{H})$ and $\overline{\mathcal{P}(G,\overline{H})} = \mathcal{P}(H,\overline{G})$ must map G onto H, showing that they are isomorphic. To see this, denote the four graphs of $\mathcal{P}(G,\overline{H})$ by G_1, $\overline{H}_2$, $\overline{H}_3$ and G_4, while the graphs of $\mathcal{P}(H,\overline{G})$ are denoted H_1, $\overline{G}_2$, $\overline{G}_3$ and H_4. Let v be a vertex in G_1; since v has degree at most $2n-1$, while all the vertices in $\overline{G}_2$ and $\overline{G}_3$ have degree at least $2n$, then without loss of generality we need only consider the case where v maps onto some vertex in H_1. Any other vertex $w \in G_1$ must map onto some vertex in H_1 or H_4; but since $d(v,w) \leq 2$, w cannot map onto a vertex in H_4. So G_1 must map entirely onto H_1.

To prove the second part of the theorem, we note that $\mathcal{P}(G,\overline{G})$ and $\mathcal{P}(H,\overline{H})$ (Figure 4.4)are always self-complementary, and they are isomorphic if and only if $G \cong H$. $\square$

$$G \text{ —— } \overline{G} \text{ —— } \overline{G} \text{ —— } G \qquad\qquad H \text{ —— } \overline{H} \text{ —— } \overline{H} \text{ —— } H$$

Figure 4.4: $\mathcal{P}(G,\overline{G})$ and $\overline{\mathcal{P}(H,\overline{H})}$

4.9. Once again, we have used the technique of substituting copies of G and H into a sc-graph. It is ironic that Colbourn and Colbourn [93] explicitly rejected the use of P_4, saying they could not find a suitable substitution, and went instead for a sc-graph on 9 vertices. Harary, Plantholt and Statman, who gave a different proof of these important results in [186], stated that whenever G and H are connected, $\mathcal{P}(G,\overline{H}) \cong \mathcal{P}(H,\overline{G}) \Leftrightarrow G \cong H$, apparently not noting that the result is true for any graphs G, H. Statman later used $\mathcal{P}(G,\overline{H})$ for his results on reconstruction (see 4.42), taking care to note that in his proof G and H would be connected. Corneil also used $\mathcal{P}(G,\overline{G})$ in showing that the Strong Perfect Graph Conjecture is reducible to the SPGC for sc-graphs, but he only needed the fact that $\mathcal{P}(G, \overline{G})$ is self-complementary (see 1.65).

We now turn to Colbourn and Colbourn's last result, showing that even the isomorphism and recognition problem for regular self-complementary graphs is equivalent to the general problem. This is not so surprising — if anything, we would expect isomorphism testing to be harder for regular graphs — but it is interesting because it shows that all graphs have a unique regular self-complementary representation. Colbourn and Colbourn actually used the self-complementary representation defined in the previous theorem to construct a regular self-complementary graph. We modify their approach slightly to do this directly. A similar construction was used by McDiarmid and Steger [248].

4.10. Theorem. *The recognition and isomorphism problems for regular self-complementary graphs is polynomially equivalent to the isomorphism problem for graphs in general.*

Proof: Given two graphs G, H, on n vertices, we form a graph $\mathcal{C}(G, \overline{H})$ as follows. We take a vertex x, two copies of G (call them G_1, G_2), and two copies of $\overline{H}$ (call them $\overline{H}_3$, $\overline{H}_4$). A vertex v_i of G is labelled v_{1i} in G_1 and v_{2i} in G_2; similarly the vertices of $\overline{H}_3$, $\overline{H}_4$ are labelled v_{3j} and v_{4j}. Each vertex of G_1 is adjacent to x and to all the vertices $\overline{H}_3$, while each vertex of G_2 is adjacent to x and to the vertices of $\overline{H}_4$. Finally,

$$\begin{aligned} v_{1i} \sim v_{2j} \quad &\Leftrightarrow \quad i \neq j \text{ and } v_i \not\sim v_j \text{ in } G \\ v_{3i} \sim v_{4j} \quad &\Leftrightarrow \quad i = j \text{ or } v_i \sim v_j \text{ in } H. \end{aligned}$$

It is not difficult to check that $\mathcal{C}(G, \overline{H})$ is regular of degree $2n$, and that if $G \cong H$ then $\mathcal{C}(G, \overline{H})$ is self-complementary. We now show that the converse is true. So let $\phi : \mathcal{C}(G, \overline{H}) \to \overline{\mathcal{C}(G, \overline{H})} = \mathcal{C}(H, \overline{G})$ be an isomorphism. Since x is the only vertex whose neighbours induce a regular graph of degree $n-1$, we must have $\phi(x) = x$, and so $\mathcal{C}(G, \overline{H}) - x \cong \mathcal{C}(H, \overline{G}) - x$. Let v be any vertex of degree $2n$ in $\mathcal{C}(G, \overline{H})$; its neighbours of degree $2n-1$ induce a subgraph isomorphic to G. Similarly, if w is any vertex of degree $2n$ in $\mathcal{C}(H, \overline{G})$, its neighbours of degree $2n-1$ induce a subgraph isomorphic to H. Thus ϕ must induce an isomorphism of G onto H.

To prove the second part of the theorem, we note that $\mathcal{C}(G, \overline{G})$ and

$\mathcal{C}(H, \overline{H})$ are always self-complementary, and they are isomorphic if and only if $G \cong H$.

Finally we note that, by 1.41, the graphs constructed will all have diameter two, so that no isomorphism test can depend on the degree and distance parameters. $\square$

Generating self-complementary graphs

4.11. If we cannot easily test a graph for self-complementarity, or check whether two sc-graphs are isomorphic, it would at least be useful to have a catalogue of self-complementary graphs. Such lists have been compiled by various authors — Alter [22] and Venkatchalam [375] for $n = 8$, Morris [256, 257] for $n = 8$ and 9, Kropar and Read [232] for $n = 12$, Faradžev [118] for $n \leq 12$, and McNally and Molina [252] for $n = 13$. The focus here is on systematically generating all self-complementary graphs, rather than providing sporadic constructions to show the existence of sc-graphs with certain properties.

The enumeration results of Chapter 7 come in handy when generating sc-graphs, as they can tell us when (or if) the total number of sc-graphs has been generated.

Counting formulas such as those of Parthasarathy and Sridharan [285] are especially useful, as instead of providing a single figure for the total number of sc-graphs on n vertices, they count the number of sc-graphs of each degree sequence. It is a common feature of most generation algorithms that they produce many duplicates, so one has to perform isomorphism checks for each graph generated. Using the Parthasarathy-Sridharan formula, we can know when the total number of non-isomorphic sc-graphs with a given degree sequence has been reached; after that, one can immediately discard any further graphs with the same degree sequence without performing an isomorphism check.

Kropar and Read [232] used this formula *after* they had generated their list of s-c graphs on 12 vertices and found that they were one graph short. They corrected their algorithm, but rather than re-generating all 720 graphs, they found the degree sequence of the missing graph, and constructed (correctly, this time) just those sc-graphs with that sequence.

Molina's method for generating odd order sc-graphs, described in 1.44, also avoids the need for many isomorphism checks, as it distinguishes between sc-graphs according to their structure. Those of type (A, B) are generated separately, and cannot possibly be isomorphic to those of type (A', B') unless $A \cong A'$ and $B \cong B'$; even the isomorphism checks between graphs of the same type must satisfy certain constraints (namely, any isomorphism would have to map A onto A' and B onto B'), and this makes for further efficiency.

Faradžev too has his own algorithm for generating various types of graphs, but the more usual methods are based on the use of antimorphisms, as described originally by Sachs [341] and Ringel [320]. The rest of this section will be devoted to their results, which show that every feasible permutation is an antimorphism of some sc-graphs, and tells us how to construct all the sc-graphs which have a given antimorphism. Ironically, Robinson [324, Section 6] and Clapham [88] used this construction to count the number of sc-graphs directly, an approach suggested by Read in his review of Sachs' paper.

4.12. Theorem. *A permutation σ is an antimorphism of some sc-graph if and only if*

A. *all the cycles of σ have length a multiple of* 4, *or*

B. *σ has one fixed vertex, all other cycles having length a multiple of* 4.

Proof: We describe an algorithm to construct all sc-graphs which have σ as an antimorphism, and show that this works only when condition A or B is satisfied. Let K_n be the complete graph on n vertices.

I. Take an arbitrary pair of vertices $\{a, b\}$, and colour the edge between them red. Colour all the edges $\sigma^{2i}\{a, b\} = \{\sigma^{2i}(a), \sigma^{2i}(b)\}$ red, and all the edges $\sigma^{2i+1}\{a, b\}$ blue, for each integer i.

II. Repeat step I for any uncoloured edges, until all edge orbits have been coloured.

III. From each orbit, choose either the red edges or the blue edges (we may choose red edges from one orbit and blue from another). The chosen edges then form a self-complementary graph.

If there are s orbits, then there are 2^s different ways of making the choices in C, though some of them may give us isomorphic graphs, and some sc-graphs may also be associated with other antimorphisms with different cycle lengths (such as σ^3, if σ's order is not a power of 2). Evidently there are no other sc-graphs with antimorphism σ, apart from the ones produced here.

The colouring described in A is well-defined, unless some edge $\{a, b\}$ is coloured both red and blue, that is, unless $\{a, b\} = \sigma^{2i+1}\{a, b\}$ for some i. This can happen in three ways:

Case 1: a and b are in the same cycle, and we have $a = \sigma^{2i+1}(a)$ and $b = \sigma^{2i+1}(b)$. Then the cycle length divides $2i + 1$, in particular, the cycle length is odd. Conversely, if there were a cycle of odd length $2i + 1 > 1$ any two vertices of the cycle would give rise to this problem, so σ can only contain even-length cycles and fixed vertices.

Case 2: a and b are in the same cycle, and we have $a = \sigma^{2i+1}(b)$ and $b = \sigma^{2i+1}(a)$. Then the cycle has length $4i + 2$. Conversely, if there were a cycle $(v_1, v_2, \ldots, v_{4i+2})$, then v_1 and v_{2i+2} would give rise to this problem, so σ cannot contain any cycles of length $2 \pmod 4$.

Case 3: a and b are in different cycles. Then we must have $a = \sigma^{2i+1}(a)$ and $b = \sigma^{2i+1}(b)$. This can only happen if both cycles have odd length; since we have ruled these out, except for the fixed vertices, we see that we cannot have two or more fixed vertices.

So the algorithm works if and only if σ has at most one fixed vertex, and all other cycles have length a multiple of 4. □

4.13. The algorithm can be modified, refined and specialised in a number of ways, which we describe below:

Self-complementary digraphs. [Zelinka 1970b] If we apply the algorithm to the complete digraph $\vec{K_n}$ we obtain all self-complementary digraphs. The only modification is that now we have ordered pairs of vertices, so (a, b) and (b, a) are distinct arcs and it does not matter if $a = \sigma^{2i+1}(b)$ and $b = \sigma^{2i+1}(a)$ (Case 2 in the proof above). Therefore, all even cycles are admissible. In the infinite case, infinite cycles are also admissible, and this is true of all the other types of self-complementary structure.

Sc-tournaments. [Zelinka 1970b, Salvi-Zagaglia 1979] A tournament cannot contain both (a, b) and (b, a), so now we must check that we do not get $a = \sigma^{2i}(b)$ and $b = \sigma^{2i}(a)$. This will happen whenever a and b are in the same cycle of length $4i$, at distance $2i$ apart. So, ironically, a permutation is the antimorphism of a sc-tournament if and only if it consists of even cycles whose length is *not* a multiple of 4, and at most one fixed vertex.

Sc-multigraphs. Instead of choosing the red edges from an orbit and discarding the blue (or *vice versa*) we can replace each red edge in an orbit by t multiple edges, and each blue edge by $r-t$ edges, for some constant r and some $t \leq r$, where we choose the value of t independently for each orbit. This will give us all self-complementary r-multigraphs, and similarly we can obtain all self-complementary r-multi-digraphs. For $r = 1$ we get the usual self-complementary graphs and digraphs.

self-converse digraphs. [Salvi-Zagaglia 1978] The construction of self-converse digraphs is significantly different. The arc orbits are now defined by $\sigma^{2i}(a,b) = (\sigma^{2i}(a), \sigma^{2i}(b))$ and $\sigma^{2i+1}(a,b) = (\sigma^{2i+1}(b), \sigma^{2i+1}(a))$. In particular:

- There is no problem if we have $a = \sigma^{2i+1}(a)$ and $b = \sigma^{2i+1}(b)$ for some i; it just means that that particular arc orbit will include both (a,b) and (b,a). So now, any cycle lengths are admissible.
- A 2-cycle of σ, say $(a\ b)$ will give us *two* separate arc orbits, (a,b) and (b,a).

For each arc orbit we can choose either to include all the arcs of that orbit, or none; of course we make each choice independently. If we make the same choice for each orbit, we obtain the trivial self-converse digraphs $\vec{K_n}$ and the null digraph.

If we want to construct self-converse r-multi-digraphs, then in each orbit we replace every arc by a bundle of t arcs, for some $t < r$, where the choice of t is made independently for each orbit. For $r = 1$ we obtain the usual self-converse digraphs as described in the previous paragraph.

4.14. Cavalieri d'Oro [65] has described a construction algorithm for self-converse digraphs as follows. Take an arbitrary digraph D with vertices $v_1, \ldots, v_n$, and a copy of its converse D' with vertices $v'_1, \ldots, v'_n$. For every i, j, we can add a pair of arcs (v_i, v'_j) and (v_j, v'_i). Finally we can add any number of vertices w_x, and any pairs of arcs of the form

- (w_x, w_y) and (w_y, w_x), or
- (w_x, v_i) and (v'_i, w_x), or

- (w_x, v_i') and (v_i, w_x).

Then the involution $(v_1v_1')(v_2v_2')\cdots(v_nv_n')(w_1)(w_2)\cdots(w_p)$ will map this digraph onto its converse. It can easily be seen that, in fact, all self-converse digraphs with a converting permutation of order 2 can be constructed in this way. However, as Salvi-Zagaglia pointed out, there are self-converse digraphs which do not have an involutory converting permutation, which is why her algorithm is necessary.

Hemminger and Klerlein [205][2] remarked that Cavalieri d'Oro's claim, that all self-converse digraphs have an involutory antimorphism, is equivalent to the claim that "All bipartite graphs with an automorphism interchanging the two parts has such an automorphism of order 2," and they presented a counterexample, due to Lovász.

To see that the two (false) claims are equivalent, we represent a digraph D with vertex-set $\{u_1, \ldots, u_n\}$ by a bipartite graph G with partition $A = \{v_1, \ldots, v_n\}$ and $B = \{w_1, \ldots, w_n\}$, in which

$$(v_i \sim w_j) \Leftrightarrow (u_i \rightarrow u_j).$$

Then the converse of D is represented by the graph G with partite sets switched, which is isomorphic to G if and only if D is self-converse.

For further exploration of this bijection between (self-converse) digraphs and bipartite graphs, using different terminology, see [31] and [164].

4.15. We can refine the basic construction algorithm in two ways. First, once we construct all sc-graphs on $4k$ vertices (with all their associated antimorphisms), we can construct the sc-graphs on $4k+1$ by adding a new vertex v and joining v either to the odd-labelled vertices or the even-labelled vertices of each cycle $(v_1v_2\ldots v_{4s})$ of the antimorphism. We can choose odd or even independently for each cycle, so if there are t cycles in the antimorphism, this will give us 2^t sc-graphs, some of them possibly isomorphic. However, Lemma 1.36 ensures that two non-isomorphic sc-graphs G and H on $4k$ vertices will always give non-isomorphic sc-graphs on $4k+1$ vertices, even if they have a common antimorphism. It is important to use all the antimorphisms for each graph G on $4k$ vertices, as each antimorphism may produce sc-graphs on $4k+1$ vertices which are not produced by other antimorphisms.

[2] The article ends with a remark about a paper in preparation by Hemminger which characterises self-converse bipartite digraphs. However, we could not find any further reference to this paper in the math reviews.

It seems that Morris [256, 257] used this approach when constructing lists of sc-graphs on 8 and 9 vertices. This technique applies equally well to self-complementary digraphs.

The second refinement is due to Gibbs [150, 151] and Salvi-Zagaglia [344], who noted that all self-complementary graphs and digraphs, and self-converse digraphs have an antimorphism of order 2^r for some r — given any antimorphism σ of order $2^r(2s+1)$ for some r, s, just consider the antimorphism σ^{2s+1}. We note that for self-complementary graphs we must have $r \geq 2$, for self-complementary digraphs we have $r \geq 1$, and for self-converse digraphs we have $r = 0$ only when the digraph is symmetric (that is, a graph with each edge replaced by a pair of opposite arcs). So we can always restrict the algorithm to antimorphisms whose cycle lengths are powers of 2 to produce a complete list of sc-graphs.

We now look at antimorphisms which have just one cycle, a subject also tackled in 1.70. (Part A of the next theorem was stated formally by Gibbs, and also by Rao [305, Observation 3.5], but it is implicit in Sachs' work).

4.16. Theorem[Sachs 1962]**.** *Let G be a self-complementary graph with a cyclic antimorphism $\sigma = (v_1 v_2 \dots v_{4k})$. Then*

A. *Each odd-labelled vertex of G is adjacent to exactly k even-labelled vertices and each even-labelled vertex is adjacent to exactly k odd-labelled vertices.*

B. *The degrees of the vertices are alternately r and $4k-1-r$, for some $k \leq r \leq 3k-1$. Moreover, for every such r, there is at least one sc-graph with antimorphism σ and degrees r and $4k-1-r$.*

Proof: Because σ^t is an antimorphism for t odd, and an automorphism for t even, we have

$$v_{2s+i} \sim v_{2s+j} \Leftrightarrow v_i \sim v_j \Leftrightarrow v_{2s+1+i} \not\sim v_{2s+1+j} \ \forall i, j, s$$

where subscripts are taken modulo $4k$. In particular, we have

$$v_1 \sim v_{2i} \Leftrightarrow v_{1+(4k+1-2i)} \not\sim v_{2i+(4k+1-2i)} \Leftrightarrow v_{4k+2-2i} \not\sim v_1.$$

Writing this out in full, we have

$$\begin{array}{rcl} v_1 \sim v_2 & \Leftrightarrow & v_1 \not\sim v_{4k} \\ v_1 \sim v_4 & \Leftrightarrow & v_1 \not\sim v_{4k-2} \\ v_1 \sim v_6 & \Leftrightarrow & v_1 \not\sim v_{4k-4} \\ & \vdots & \\ v_1 \sim v_{2k} & \Leftrightarrow & v_1 \not\sim v_{2k+2}. \end{array}$$

So v_1 is adjacent to exactly half of the $2k$ even-labelled vertices. Exactly similar arguments hold for every other vertex, and this proves part A. It also shows that the degree of v_1, say r, must be at least k and at most $3k-1$. Obviously $\sigma(v_1) = v_2$ will have degree r in $\sigma(G) = \overline{G}$, and thus v_2 has degree $4k+1-r$ in G, and the degrees will alternate like this between r and $4k-1-r$. This proves the first part of B.

We now consider the adjacencies of v_1 with the odd-labelled vertices. We have

$$v_1 \sim v_{2i+1} \Leftrightarrow v_{1+(4k-2i)} \sim v_{2i+1+(4k-2i)} \Leftrightarrow v_{4k+1-2i} \sim v_1.$$

Again, writing this out in full we have

$$\begin{array}{rcl} v_1 \sim v_3 & \Leftrightarrow & v_1 \sim v_{4k-1} \\ v_1 \sim v_5 & \Leftrightarrow & v_1 \sim v_{4k-3} \\ v_1 \sim v_7 & \Leftrightarrow & v_1 \sim v_{4k-5} \\ & \vdots & \\ v_1 \sim v_{2k-1} & \Leftrightarrow & v_1 \sim v_{2k+3}. \end{array}$$

So $2(k-1)$ of the adjacencies with the odd-labelled vertices must occur in pairs. Finally, the adjacency of v_1 with v_{2k+1} can be chosen independently of the other adjacencies of v_1.

So if we want to construct a sc-graph with v_1 having even degree $r = k+2s$ where $0 \leq s \leq k-1$, we choose k adjacencies with the even-labelled vertices, and exactly $2s$ adjacencies with the odd-labelled vertices. If we want v_1 to have odd degree $r = k + 2s + 1$, we also choose the adjacency of v_1 with v_{2k+1}. □

4.17. Theorem[Sachs 1962]**.** *Every antimorphism is the antimorphism of some regular or almost regular sc-graph.*

Proof: We first show, by induction on the number of cycles, that an antimorphism σ on $4k$ vertices has an associated almost regular sc-graph. When

σ consists of just one cycle, the result follows from the previous theorem by putting $r = 2k-1$. If σ has $t > 1$ cycles, then by induction we can construct an almost regular sc-graph A on the first $s-1$ cycles, and another almost regular sc-graph B on the last cycle.

If $|V(A)| = 4a$ and $|V(B)| = 4b$, where $a+b = k$, then A will have degrees $2a-1$ and $2a$, while B will have degrees $2b-1$ and $2b$. We now show how to join every vertex of A to exactly $2b$ vertices of B, and every vertex of B to exactly $2a$ vertices of A to construct a sc-graph with degrees $2k-1$ and $2k$.

Let the vertices of each cycle be numbered consecutively, and let A_i [resp. B_i] be the set of vertices of A [resp. B] with subscripts congruent to $i \pmod 4$. We note that $|A_i| = a$, $|B_i| = b$, $\sigma(A_i) = A_{i+1}$ and $\sigma(B_i) = B_{i+1}$ for all i, with subscripts taken modulo 4.

If we join the vertices of A_1 to those of B_1 and B_2, those of A_2 with those of B_1 and B_4, those of A_3 with those of B_3 and B_4, and the vertices of A_4 with those of B_2 and B_3, it can be checked that the resulting graph is self-complementary and, as noted above, almost regular.

Now, if we have a permutation $\sigma' = (v_0)\sigma$ on $4k+1$ vertices, we construct an almost regular sc-graph on σ as above, and then join v_0 to the vertices of degree $2k-1$, thus obtaining a regular sc-graph with antimorphism σ'. $\square$

4.18. We note that this is very similar to the method we used in 1.28 to construct sc-graphs of diameter 3 without end-vertices. Results on the antimorphisms of self-complementary hypergraphs and almost self-complementary graphs can be found in 5.1 and 5.41, respectively.

Self-complement indexes

4.19. It is interesting to measure how "close" a graph is to being self-complementary. There are a couple of intuitive ways of doing this, which we will describe below. The *self-complement index* $s(G)$ is defined as the order of the largest induced subgraph H of G such that $\overline{H}$ is also induced in G. For a graph of order n we have $1 \le s(G) \le n$, and the following result is easy to prove:

Lemma[Akiyama, Exoo and Harary 1980]**.** *Let G be a graph of order n. Then*

A. $s(G) = s(\overline{G})$.

B. $s(G) = n$ *if and only if* G *is self-complementary.*

C. If H *is an induced self-complementary subgraph of* G *then* $s(G) \geq |V(H)|$. $\square$

4.20. The *induced number* $m(G)$ is the minimum order of a graph which contains both G and $\overline{G}$ as induced subgraphs; obviously $m(G) \geq n$, and A and B above are also true for $m(G)$. Akiyama *et al.* showed that in fact the two indexes are equivalent.

Theorem. *If* G *is a graph of order* n *then* $m(G) = 2n - s(G)$.

Proof: Let G have n vertices, and let H of order s be a largest induced subgraph of G whose complement is also induced in G. Take a copy of G and a copy of $\overline{G}$; these will both contain H as an induced subgraph. By "superimposing" these copies of H we obtain a graph of order $2n - s$ in which both G and $\overline{G}$ are induced subgraphs, so that $m(G) \leq 2n - s$.

Now let F be a graph in which G and $\overline{G}$ are both induced subgraphs. Let X [resp. Y] be sets of vertices inducing G [resp. $\overline{G}$]. Then the subgraph induced by $X \cap Y$ is an induced subgraph of both G and $\overline{G}$, so by 4.19.C $|X \cap Y| \leq s$. Thus $m \geq |X \cup Y| \geq 2n - s$. $\square$

4.21. Theorem. *For all* n*, and all positive integers* $k < n$*, there is a graph* G *of order* n *with* $s(G) = k$*. A graph* G *with* $s(G) = n$ *exists if and only if* $n \equiv 0$ *or* $1 \pmod 4$.

Proof: The second part of the theorem is obvious. We now prove the first part.

Case A. $k \equiv 0$ or $1 \pmod 4$. Let H be a sc-graph of order k, and define $G := H \cup \overline{K}_{n-k}$; then $s(G) = k$.

Case B. $k \equiv 2$ or $3 \pmod 4$. Let H be a sc-graph of order $k - 2$, and construct a graph G as follows: add two new vertices u, v, joining them to all the vertices of H (but not to each other), and a third vertex w which we join to v; finally add $n - k - 1$ isolated vertices (see Figure 4.5). Then the graphs induced by $V(H) \cup \{v, w\}$ and $V(H) \cup \{u, v\}$ are complements of each other, so that $s(G) \geq k$. Any larger induced subgraph must be either

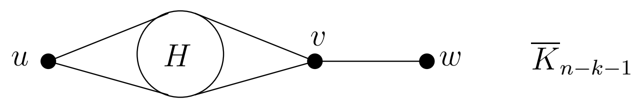


Figure 4.5: A graph with $s(G) = k$, $k \equiv 2$ or $3 \pmod 4$

the graph induced by $V(H) \cup \{u, v, w\}$, whose complement is not induced in G; or else must contain an isolated vertex, so that its complement contains a vertex of degree at least k, which does not occur in G. □

4.22. We introduce some notation before the next result. A *pendant edge* of G is an edge incident to an endvertex. A complete graph on n vertices, with one pendant edge attached, is denoted by $K_n \cdot K_2$; while $K_n + K_2 \circ K_1$ denotes a complete graph on $n+2$ vertices, with two independent pendant edges (see 1.10).

Theorem. *The self-complement indexes of complete graphs, complete bigraphs, complete graphs plus one pendant edge, trees, cycles, unicyclic graphs, and complete graphs with two independent pendant edges, are given by*

A. $s(K_n) = 1$*;*

B. $s(K_{m,n}) = 2$*, for* $\max(m, n) \geq 2$*;*

C. $s(K_n \cdot K_2) = 3$*;*

D. $s(C_n) = 4$*, for* $n \geq 6$

E. $s(T) = 4$ *when* T *is a tree that is not a star;*

F. $s(K_n + K_2 \circ K_1) = 5$. □

The proof of D simply notes that for any graph G with five or more vertices, either $G = \overline{G}$ is a pentagon, or else G or $\overline{G}$ contains a triangle. It then follows that $s(G) = 4$ for any unicyclic graph with cycle of length at least 6; $s(G) = 5$ for a unicyclic graph with cycle of length exactly 5; and $s(G) = 4$ for any unicyclic graph containing C_4, except for C_4 itself, which has self-complement index 2.

4.23. Another measure of self-complementarity is the self-complementary closure number, $\mathrm{sc}(G)$, the order of the smallest self-complementary graph which contains G as an induced subgraph.

Theorem. *For any graph G of order n*

A. $sc(G) = sc(\overline{G})$.

B. $sc(G) \geq m(G) \geq n$, *and* $sc(G) = n$ *if and only if* G *is self-comlementary.*

C. $sc(G) \leq 4n$.

D. $sc(G) \leq 2n$ *if* G *has a pair of interchangeable sets.*

Proof: A and B are obvious. For C, consider the P_4-join of $(G, \overline{G}, \overline{G}, G)$, for example. D was proved by Nair [266] (see 1.42 for the proof) and Rall [297]. □

Rall also showed that for bipartite graphs, $\mathrm{sc}(G) \geq 2n - 4$.

4.24. It would be interesting to consider the size of the largest induced self-complementary subgraph of G, a parameter which we can denote by $\mathrm{si}(G)$. This has the usual properties one would like it to have:

Lemma. *For a graph G of order n*

A. $si(G) = si\overline{G}$.

B. $1 \leq s(G) \leq si(G) \leq n$, *and* $si(G) = n$ *iff* G *is self-complementary.*

C. *For all* $k \leq n$, $k \equiv 0$ *or* $1 \pmod 4$, *there is a graph of order* n *with* $si(G) = k$. □

However, there is no obvious equivalence between $\mathrm{sc}(G)$ and $\mathrm{si}(G)$, as there was for $s(G)$ and $m(G)$.

4.25. The following results, due to Benhocine and Wojda [36], are of related interest, but here the graphs and digraphs are *not* induced subgraphs of self-complementary graphs or digraphs.

Theorem. *Every digraph with $n \geq 3$ vertices and at most n arcs is contained in a sc-digraph of order n.* $\square$

4.26. Theorem. *Let G be a graph of order $n \equiv 0$ or $1 \pmod 4$, with at most $n-1$ edges. Then G is contained in a sc-graph S of order n, unless $G \in \{C_4 \cup K_1, C_3 \cup K_{1,n-4}, K_{1,n-1}\}$. Moreover, S has an antimorphism of order 4.* $\square$

4.27. Conjecture[Benhocine and Wojda 1985]**.** *Every digraph of order n with at most $2n-3$ edges is contained in a sc-digraph of order n, unless n is even and D (or its converse) is the digraph with arcs*

$$(v_1, v_2), (v_1, v_3), \ldots, (v_1, v_n), (v_{n-1}, v_n), (v_{n-2}, v_1), (v_{n-3}, v_1), \ldots, (v_2, v_1).$$

4.28. Chartrand *et al.* [72] have defined another measure of self-complementarity, one which is not as obvious as the ones considered so far. Let G be any graph, H a set of its vertices, and $H' := V(G) - H$. Then G *switched on* H, denoted by $S_H(G)$ is the graph with $V(S_H(G)) = V(G)$ and

$$E(S_H(G) = E(\overline{H}) \cup E(\overline{H, H'}) \cup E(H')$$

where $E(\overline{H, H'})$ denotes the set of edges between H and H' which are *not* present in G. When H is just a single vertex, we get the usual switching operation; while $S_\emptyset(G) = G$ and $S_{V(G)}(G) = \overline{G}$. So $S_H(G)$ is intermediate between G and $\overline{G}$. The switching number $\text{sw}(G)$ is then defined to be the largest number of vertices in a set H for which $S_H(G) \cong G$.

Lemma. *A. For any graph G and any set of vertices H*

$$\overline{S_H(G)} = S_H(\overline{G}) \text{ and thus } \text{sw}(G) = \text{sw}(\overline{G}).$$

B. *For a graph G of order n, $0 \leq \text{sw}(G) \leq n$, and $\text{sw}(G) = n$ if and only if G is self-complementary.*

C. *For all graphs G of order n, $\text{sw}(G) \neq n-1$.*

D. *If G is a regular graph of order n and degree $r \neq \frac{n-1}{2}$, then $\text{sw}(G) = 0$. In particular*

$$\text{sw}(K_n) = \text{sw}(\overline{K}_n) = 0 \text{ for } n \geq 2, \text{ and}$$

$$sw(C_n) = 0 \text{ for } n \neq 5.$$

Proof: A and B are easy. For C, we note that if H is a set of $n-1$ vertices, then $S_H(G) = S_{V(G)}(G)$. For D we note that, for any set $\emptyset \neq H \subseteq V(G)$, where $S_H(G) \cong G$, and any $v \in H$,

$$d_G(v) = r = n - r - 1 = d_{S_H(G)}(v),$$

so we must have $r = \frac{n-1}{2}$. □

4.29. Part C of the lemma above gives us a hint that sw(G) might have some strange behaviour. In fact, Chartrand *et al.* showed that there do not always exist graphs G of order n with sw$(G) = k$. The constraints on n and k, in terms of their residue (mod 4), are shown in Table 4.1. The question marks in the table denote combinations of n and k for which the existence is still open.

k \ n	0	1	2	3
0	Yes	Yes	Yes	Yes
1	No	Yes	No	Yes
2	No	No	No	No
3	?	No	?	No

Table 4.1: The existence of a graph G of order n (mod 4) with sw$(G) = k$ (mod 4).

There are still graphs which are arbitrarily close (or far away) from being self-complementary, in the following sense: If a graph G of order n has switching number k, define its switching coefficient to be $\frac{k}{n}$. Then every rational number number $0 \leq r \leq 1$ is a switching coefficient of an arbitrarily large graph. For $r = 0$ take any of the regular graphs described in the lemma above, and for $r = 1$ take any self-complementary graph. For $r = \frac{a}{b}$, take a graph with $4mb$ vertices and switching number $4ma$, for some m; the existence of such a graph is guaranteed by Chartrand *et al.*'s results.

4.30. It might seem that the switching number is not as interesting as the other measures of self-complementarity defined previously, because of the non-existence results summarised in Table 4.1. But we must not forget that self-complementary graphs themselves only exist when $n \equiv 0$ or $1 \pmod 4$. This particular gap is filled by the almost self-complementary graphs (asc-graphs), which exist only for $n \equiv 2$ or $3 \pmod 4$; a graph G is almost self-complementary if it is isomorphic to $\tilde{K}_n - G$, where $\tilde{K}_n = K_n - e$ for some edge e of K_n. Since $\mathrm{sw}(G)$ is equal to n precisely for self-complementary graphs, and can never be equal to $n-1$, the following result is quite appropriate.

Proposition. *For a graph G on n vertices, $sw(G) = n-2$ if and only if G or $\overline{G}$ is almost self-complementary.*

Proof: Let H be a set of $n-2$ vertices such that $S_H(G) \cong G$, and let $V(G) - H = \{u, v\}$.

If $e = uv$ is not an edge of G, then G and $S_H(G)$ are complements in $K_n - e$, and so G is an asc-graph. If $e = uv$ *is* an edge of G, then it will not be an edge of $\overline{G}$, and since $S_H(\overline{G}) = \overline{S_H(G)} \cong \overline{G}$, the previous argument shows that $\overline{G}$ is almost self-complementary.

Conversely, any graph G which is self-complementary in $K_n - e$ will have $S_H(G) \cong G$, and since it must have $n \equiv 2$ or $3 \pmod 4$, it cannot be self-complementary; therefore $\mathrm{sw}(G)$ is exactly $n-2$. Also, $\overline{G}$ will satisfy $S_H(\overline{G}) = \overline{S_H(G)} \cong \overline{G}$, so that $\mathrm{sw}(\overline{G}) = n-2$. $\square$

4.31. In his study of almost self-complementary graphs, Clapham [89] noted that their self-complement index is $n-1$, but the converse is not true — there are graphs with $s(G) = n-1$ even for $n \equiv 0$ or $1 \pmod 4$. Furthermore, for $n \equiv 2$ or $3 \pmod 4$ (and thus $n-1 \equiv 1$ or $2 \pmod 4$) we can take the examples of Theorem 4.21, which are not asc-graphs. For further details see 5.35–5.44; the result on antimorphisms of asc-graphs shows that they may contain induced sc-subgraphs on $n-2$ vertices, but then again, they may not. We therefore propose the following line of enquiry:

Problem. *What can we say about $si(G)$ when G is almost self-complementary?*

Logical connections

4.32. Many of the results on self-complementary graphs are really corollaries of theorems about complementary graphs, of the form "For every graph G with at least N vertices, property P holds either in G or in $\overline{G}$".

Such results (usually called Nordhaus-Gaddum-type results when they involve some invariant like chromatic number — see 1.57) have been investigated by McKee. In graph theory we are obviously interested in *valid* properties, which are true of all graphs. A property which is true for every graph or its complement, is said to be *semi-valid*. The existence of self-complementary graphs rules out the possibility of any property which is valid in exactly one of G and $\overline{G}$ for all G.

The best-known semi-valid property is connectivity — every graph is either connected or has a connected complement (1.8). Even easier are the properties "has at least $\frac{1}{2}\binom{n}{2}$ edges" and "has at most $\frac{1}{2}\binom{n}{2}$ edges". Evidently, any property that follows from a semi-valid property is itself semi-valid.

Many properties are semi-valid only for graphs with enough vertices; for example "contains a triangle" holds for graphs of order at least 6. We can say that this property is "eventually semi-valid"; to express it rigorously we have to put it as "has order less than six or contains a triangle." Similarly we have "has order less than nine or is nonplanar" (1.46), and "has order less than six or is both connected and has a pancyclic line graph" (2.23).

If we assume a language strong enough to include "is isomorphic to G" as a sentence, for a given graph G, then we have the following logical characterisation.

Lemma. *A graph is self-complementary if and only if it satisfies all semi-valid sentences.*

Proof: Obviously a sc-graph satisfies all semi-valid sentences. Now, let G be a non-self-complementary graph and let σ be the sentence "is not isomorphic to G". Then σ is semi-valid, but G does not satisfy σ. $\square$

For a detailed treatment see [250], and see [251] for an extension to the case where the complete graph is split into three subgraphs (instead of two, as happens with a graph and its complement).

Incidentally, for a given property P and a graph G which has that property, Chartrand *et al.* proposed the question of determining the size of the largest H for which $S_H(G)$ also has P (see 4.28 for definitions). We might even try to find all subsets H for which $S_H(G)$ has property P. Given a particular collection of subsets $\emptyset \neq H \neq V(G)$, which are the graphs for which $G = S_\emptyset(G)$, $S_H(G)$ and $S_{V(G)}G = \overline{G}$ all have property P? Just as switching a graph on H can be seen as a partial complementation, so we can say that these problems are about partial semi-validity.

4.33. Another logical approach of relevance to self-complementarity is the first-order logical theory of graphs developed by Blass and Harary in [43]. Their first-order language L consists of predicate symbols for equality and adjacency, propositional connectives *not*, *and*, and *or*, the *existential quantifier* $\exists$, and the propositional constants *true* and *false*. The axioms of the theory include *Axiom* 0, which stipulates that a graph has at least two points and that adjacency is irreflexive and symmetric. For $k \geq 1$, *Axiom* k is satisfied by a graph if, for any sequence of $2k$ of its points, there is another point adjacent to the first k points but not to the last k. Blass and Harary proved that any L-sentence which holds for almost all graphs is deducible from finitely many of the axioms.

In [44] they discovered the first family of graphs known to satisfy Axiom k for each k; it turns out that this family is just the Paley graphs. They then used these results to show that self-complementarity cannot be expressed as an L-sentence.

Theorem. *If* $p \equiv 1 \pmod 4$ *is prime, and* $p > k^2 2^{4k}$, *then the Paley graph on* p *vertices satisfies Axiom* k. *Consequently, the following properties of graphs are not first order:*

A. self-complementarity

B. regularity

C. Eulerian-ness

D. rigidity (lack of non-trivial automorphisms)

Proof: We will prove only the second part. If P is any of the first three properties then we cannot deduce "not P" from finitely many axioms, because for any finite set of axioms there is a Paley graph which satisfies P and all of the axioms. We also know that "not P" holds for almost all graphs (see 7.15 for self-complementarity), and so "not P" cannot be an L-sentence; therefore P cannot be an L-sentence. For D, the argument is the same except that P and "not P" change roles. $\square$

The theorem does not really tell us much about how easy or difficult it is to recognise self-complementary graphs, because Eulerian-ness and regularity, which are both very easy to recognise, are not first order either.

The bound given above on the order of Paley graphs satisfying Axiom k is quite rough. For example, the theorem tells us that the Paley graphs on 17 and 1033 vertices satisfy Axioms 1 and 2, respectively, but it is known that so do the Paley graphs on 5 and 61 vertices, respectively. (In fact the pentagon is the smallest graph to satisfy Axiom 1). It is an unsolved problem to find the minimum order of a graph satisfying Axiom k, and the least p for which the Paley graph on p vertices satisfies Axiom k.

The reconstruction conjectures

4.34. The deck $\mathcal{D}(G)$ of a graph G consists of the multi-set of vertex-deleted subgraphs of G, that is, the collection of induced subgraphs H with $|V(H)| = |V(G)|-1$. The edge-deck $\mathcal{ED}(G)$ consists of the set of subgraphs H (not necessarily induced) with $|E(H)| = |E(G)| - 1$. Note that all subgraphs are unlabelled, and may appear more than once in the deck. Two decks or edge-decks $(H_1, \ldots, H_r), (H'_1, \ldots, H'_r)$ are said to be isomorphic if there is a permutation ϕ such that $H_i \cong H'_{\phi(i)} \forall i$. The Graph Reconstruction Conjecture [Edge-Reconstruction Conjecture] states that two graphs with at least 3 vertices [at least 4 edges] are isomorphic if and only if they have isomorphic decks [edge-decks]. A graph whose [edge-]deck is unique up to isomorphism is said to be [edge-]reconstructible.

It is known that disconnected graphs and regular graphs are reconstructible, and that (in a probabilistic sense) almost all graphs are reconstructible. It is also known that the Reconstruction Conjecture implies the Edge Reconstruction Conjecture, but both of them remain till today among the most important open problems in graph theory.

4.35. Clapham and Sheehan's investigations [91] into the edge-reconstruction conjecture have included a look at self-complementary graphs, but without any concrete results in this regard.

Lovász [240] showed that graphs with $|E(G)| > \frac{1}{2}\binom{n}{2}$ are edge-reconstructible. For us this is still "an edge too far", but it comes in handy in the following case.

Let the i-th power of G be the graph G^i, with the same vertices as G and an edge between any two vertices at distance at most i in G. Obviously when i is greater than or equal to the diameter, G^i is just K_n. So for self-complementary graphs the only interesting case is when $i = 2$ and G has diameter 3. Since G^2 has more edges than G, Lovász' result settles the matter completely.

For sc-graphs themselves, Müller [259] comes to our rescue, by showing that a graph with n vertices and m edges is edge-reconstructible if $2^{m-1} > n!$. This means that the edge-reconstruction conjecture is true for all graphs with $|E(G)| \geq \frac{1}{2}\binom{n}{2}$ and $n \geq 12$. Now McKay [249] showed that all graphs with at most 11 vertices are reconstructible, and since all reconstructible graphs are edge-reconstructible, his result fits nicely with that of Müller and Lovász to give us:

Theorem. *All graphs with at least $n(n-1)/4$ edges are edge-reconstructible. In particular all sc-graphs and their squares are edge-reconstructible.* □

4.36. What about the reconstruction conjecture for digraphs, hypergraphs or infinite graphs? All of these have been shown to be false, in the sense that there are infinite families of non-reconstructible digraphs, hypergraphs and infinite graphs. The digraph counterexamples, found by Stockmeyer [363, 364], are especially interesting to us because they are either sc-tournaments, or based on sc-tournaments. We describe them briefly here.

Definition. We say that a vertex v_i dominates vertex v_j, written $v_i \to v_j$, if there is an arc directed from v_i to v_j. A tournament is a digraph with vertices $\{v_1, \ldots, v_n\}$ in which, for any $i \neq j$, either $v_i \to v_j$ or $v_j \to v_i$, but not both. The score of v_i is the number of vertices that it dominates. We call v_i an odd [even] vertex if i is odd [even].

Definition. Any integer can be written uniquely as $2^r s$, for some r, s, where s is an odd integer; we define $\mathrm{pow}(2^r s) = r$ and $\mathrm{odd}(2^r s) = s$. In particular, $\mathrm{pow}(-1) = 0$ and $\mathrm{odd}(-1) = -1$.

Definition. For each integer of the form $n = 2^k$, T_n is the tournament with vertices $\{v_1, \ldots, v_n\}$ where, for $i \neq j$,

$$v_i \to v_j \text{ iff } \operatorname{odd}(j-i) \equiv 1 \pmod 4.$$

Since $\operatorname{odd}(j-i) = -\operatorname{odd}(i-j)$, this rule defines exactly one arc between v_i and v_j. We now list the properties that make these tournaments interesting. For proofs, we refer to [363].

4.37. Theorem. *For each integer $n = 2^k$ the following are true:*

A. *T_n is a sc-tournament, as the mapping $\psi : v_i \leftarrow v_{n+1-i}$ reverses the direction of all arcs.*

B. *The first $n/2$ vertices each have score $n/2$, while the remaining $n/2$ each have score $n/2 - 1$.*

C. *The first $n/2$ vertices induce a copy of $T_{n/2}$, as do the last $n/2$ vertices.*

D. *T_n has only the identity automorphism.*

E. *For each integer i, with $1 \leq i \leq n$, the tournaments $T_n - v_i$ and $T_n - v_{n+1-i}$ are isomorphic. Moreover, there is an isomorphism between them mapping all odd vertices onto even vertices, and vice versa.*

F. *Since $T_n - v_i \cong T_n - v_{n+1-i}$, and (by A) $T_n - v_{n+1-i} \cong T_n' - v_i$, each vertex-deleted subtournament of T_n is a sc-tournament.* □

4.38. It is clear from A that the two sets of odd and even vertices are exchangeable in the sense of 1.37. We can thus form two non-isomorphic sc-tournaments by adding a vertex v_0; in A_{n+1} the vertex v_0 dominates the odd vertices and is dominated by the even vertices, while in B_{n+1} the vertex v_0 dominates the even vertices and is dominated by the odd vertices.

It is clear that $A_{n+1} - v_0 \cong T_n \cong B_{n+1} - v_0$, and that any isomorphism satisfying E can be extended to an ismorphism from $A_{n+1} - v_i$ onto $B_{n+1} - v_{n+1-i}$ for each integer i with $1 \leq i \leq n$. Thus $\mathcal{D}(A_{n+1}) \cong \mathcal{D}(B_{n+1})$; Stockmeyer then proved that $A_{n+1} \ncong B_{n+1}$, and therefore they form a non-reconstructible pair of tournaments.

We note that the first $n/2$ vertices and the last $n/2$ vertices of T_n also form exchangeable sets; we can thus form two sc-tournaments by adding a vertex v_0 which either dominates or is dominated by precisely the first $n/2$ vertices. However, these tournaments will have different score sequences and so, by Manvel [244], cannot have isomorphic decks.

In [364] Stockmeyer went much further, using the tournaments T_n to build six non-reconstructible pairs of digraphs (including one pair of tournaments) for each integer of the form $n = 2^k + 2^m$, with $0 \leq m < k$, $k \geq 2$; and three non-reconstructible pairs of digraphs for $n = 2^k$. He noted that all known non-reconstructible tournaments of odd order are self-complementary, while the known even order counterexamples form complementary pairs; however, the non-reconstructible digraphs do not show this pattern. Besides, each of Stockmeyer's counterexamples shows that Manvel's result on degree-pair sequences, cited above, cannot be extended to give the degree-pair of each missing vertex individually.

4.39. Not only are Stockmeyer's tournaments B_n and C_n self-complementary, they also have a vertex-deleted sc-subtournament, each of whose subtournaments are in turn also self-complementary. This is interesting because of results obtained by other authors, which we describe below.

Definition. A transitive finite tournament has vertices $\{v_1, \ldots, v_n\}$ and $v_i \to v_j$ iff $i > j$. The vertices v_1 and v_n are called the source and sink, respectively. An almost transitive tournament is obtained from a transitive tournament by reversing the arc joining sink to source.

We note that an infinite transitive tournament can be defined as above on the vertex set $\{v_1, \ldots\}$; in this case, it will have a source but no sink. However, we could also define a tournament on vertex-set $\{v_1, \ldots\} \cup \{w_1, \ldots\}$, with v_1 as source and w_1 as sink, the arcs being

$$\begin{array}{ccc} v_i \to v_j & \text{iff} & i > j \\ v_i \to w_j & \text{for all} & i, j \\ w_i \to w_j & \text{iff} & i < j. \end{array}$$

We can then form an infinite almost-transitive tournament by reversing the arc from v_1 to w_1.

Reid and Thomassen [319] characterised the *strongly self-complementary tournaments*, those tournaments for which every sub-tournament is self-complementary. They showed that these are just the finite transitive and almost-transitive tournaments, and five tournaments of order 5, 6 and 7.

Trotter (unpublished) defined two tournaments T, U, on the same vertices to be *hereditarily isomorphic* if any subset of the vertices induces two isomorphic subtournaments. Reid and Thomassen characterise all hereditarily isomorphic tournaments, showing in particular that the strongly self-complementary tournaments, and the infinite transitive and almost-transitive tournaments, are the only hereditarily isomorphic tournaments such that $T - U$ is a spanning connected subgraph.

Let R_1, R_2 be two tournaments on the same set of n vertices, $n \geq 13$ or infinite. Hagendorf and Lopez [169] showed that if, for any proper subset S of the vertices, the induced sub-tournaments $R_1[S]$ and $R_2[S]$ are either isomorphic or complementary, then R_1 and R_2 are also isomorphic or complementary, and in fact the pairs of induced subtournaments are either all isomorphic or all complementary.

Boudabbous and Boussairi [46] defined a tournament to be *decomposable* if its vertex-set can be partitioned into A_1, A_2, with $1 < |A_i| < n$, such that every vertex in A_1 dominates all the vertices of A_2. They showed that for a tournament T, if all sub-tournaments on $n-3$ vertices are decomposable and self-complementary then

A. if $n \geq 12$ then T is transitive or almost-transitive

B. if $n \geq 14$ then T can be reconstructed from its $n-3$ subtournaments.

4.40. Kocay [224] described a novel way of treating the deck of a graph using hypergraphs. Given a 3-uniform hypergraph H, and a vertex $i \in V(H)$, the set of hyperedges containing i define a graph H_i with $V(H_i) = V(H) - i$ and $E(H_i) = \{j,k\}|\{i,j,k\} \in E(H)\}$. We say that H_i is subsumed by H. There are two particularly interesting cases. The first is when the subsumed graphs are all isomorphic, say $H_i \cong G$ for some fixed graph G, we say that H subsumes G.

The second interesting case is when a set of graphs G_i form the set of subsumed graphs of a hypergraph H, and also the deck of a graph G; that is $H_i \cong G_i \cong G - i$ for each $i \in V(G) = V(H)$. In this case we say that H subsumes $\mathcal{D}(G)$. We note that not all graphs are subsumed or have their decks subsumed; and those which do may be subsumed by several non-isomorphic hypergraphs. In any case, whether G or $\mathcal{D}(G)$ is subsumed does not tell us whether G is reconstructible.

One of the limitations of this concept is that it does not apply to graphs with the 'wrong' number of vertices or edges. Since every edge $\{i, j, k\}$ of a hypergraph H will appear in the subsumed graphs H_i, H_j and H_k, we have $|E(H)| = \frac{1}{3}\sum |E(H_i)|$. And since every edge $\{v, w\}$ of a graph will appear in all vertex-deleted subgraphs except $G - v$ and $G - w$, we have $|E(G)| = \frac{1}{n-2}\sum |E(G - i)|$ (Kelly's Lemma). So for H to subsume $\mathcal{D}(G)$ we must have $|E(H)| = \frac{n-2}{3}|E(G)|$, and thus either $n \equiv 2 \pmod 3$ or $|E(H)| \equiv 0 \pmod 3$. It can be checked, however, that all sc-graphs satisfy these criteria.

By studying the Galois field $GF(n)$ and its group of linear transformations, Kocay constructed a family of vertex-transitive, self-complementary hypergraphs on $4k + 1$ vertices. By vertex-transitivity, the subsumed graphs of each hypergraph are necessarily all isomorphic, say to some graph G on $4k$ vertices. It turns out that G is also self-complementary, its vertices form exactly two orbits, and its deck is also subsumed by a hypergraph. Kocay also showed that the Paley graphs P_n are subsumed by a hypergraph; and if $P_n - v$ is a vertex-deleted subgraph of P_n, then $P_n - v$ and $\mathcal{D}(P_n - v)$ are also subsumed by hypergraphs.

4.41. We have now seen quite a bit of the "non-standard" part of reconstruction, and self-complementary graphs have not always been on the right side. It is therefore good to end with another non-standard aspect of reconstruction, in which sc-graphs might end up playing the role of the knight in shining armour. Statman [362] showed that the Graph Reconstruction Conjecture is equivalent to a statement concerning self-complementary graphs. If it can be proved that certain conditions imply self-complementarity, the reconstruction conjecture for graphs would be true. It is a long shot, but Statman's work is interesting and we reproduce the relevant parts here.

Given a graph G on n vertices, the *n-permutation tree* T_G is a directed rooted tree with all arcs directed away from the root. The root is labelled G and is joined to nodes labelled $G - v_1, G - v_2, \ldots, G - v_n$. In general each node labelled with a subgraph $G - [v_i, \ldots, v_j]$ is joined to nodes labelled $G - [v_i, \ldots, v_j, v_k]$ for each $v_k \notin [v_i, \ldots, v_j]$. Note that the vertex sequences are ordered, so there will be many nodes corresponding to the same subgraph of G. We create $n - 2$ levels (apart from the root), so that the labels on the leaves correspond to pairs of vertices of G. Each leaf is then coloured 1 or 0 depending on whether there is an edge or non-edge between the corresponding pair of vertices.

Two trees are said to be isomorphic if there is a bijection from one to the other preserving adjacencies and colours. It is obvious that an isomorphism between two graphs extends naturally to an isomorphism between their trees. If, conversely, $T_G \cong T_H \Rightarrow G \cong H$, then we say that a G is *recognizable from its tree*.

Note that, while we can use the labels to help define an isomorphism, there is no need to do so, nor to preserve labels. For example, let us say an isomorphism from T_{G_1} to T_{G_2} maps a node labelled $G_1 - [u_1, v_1]$ onto a node labelled $G_2 - [u_2, v_2]$. To preserve adjacencies, any node labelled $G_1 - [u_1, v_1, w]$ must be mapped onto a node labelled $G_2 - [u_2, v_2, z]$ for some z. But the node labelled $G_1 - [v_1, u_1]$ could very well be mapped onto $G_2 - [x_2, y_2]$, where $x_2 \neq v_2$ and $y_2 \neq u_2$.

Theorem. *The following are equivalent:*

A. *All graphs are reconstructible from their decks.*

B. *All graphs are recognizable from their trees.*

Proof: Let B be true. Then given the deck of a graph G we can construct the trees $T_{G-v_1}, T_{G-v_2}, \ldots, T_{G-v_n}$. If we now create a root node and join it to the roots of the n trees, we obtain T_G, and thus uniquely identify G.

Now let A be true. We prove B by induction on n. A 2-permutation tree T_G has a single node which is coloured 1 iff $G \cong K_2$ and 0 iff $G \cong \overline{K}_2$. So let any graph on $n \geq 2$ vertices be recognizable from its tree. Then if we are given an $n+1$-permutation tree T_G, each n-permutation sub-tree will identify a subgraph $G - v_i$ of G up to isomorphism. We can thus find the deck of G and, by A, reconstruct G uniquely. $\square$

Note that the theorem does not say that if a specific graph G is reconstructible then it is also recognizable from its tree. To recognize a graph we have to reconstruct recursively the deck of G before we can reconstruct G itself. So the reconstruction conjecture would have to be true for all the induced subgraphs of G.

4.42. Theorem. *The following statements are equivalent:*

A. The Graph Reconstruction Conjecture is true.

B. A graph G is self-complementary if and only if $T_G \cong T_{\overline{G}}$.

Proof: If A is true, then B follows from Theorem 4.41. We will show that if A is false, then there is a graph $\mathcal{P} \ncong \overline{\mathcal{P}}$ such that $T_{\mathcal{P}} \cong T_{\overline{\mathcal{P}}}$.

If the graph reconstruction conjecture is false, then by Theorem 4.41 there are two graphs $G \ncong H$ such that $T_G \cong T_H$. Note that if $\tau : T_G \cong T_H$ is an isomorphism; then τ^{-1} will also be an isomorphism from $T_{\overline{H}}$ onto $T_{\overline{G}}$. Moreover, τ induces an obvious bijection from ordered sequences of vertices of G, $[u_1, \ldots, u_i]$, onto ordered sequences of vertices of H.

Let $\mathcal{P} = \mathcal{P}(G, \overline{H})$ be the P_4-join of $(G, \overline{H}, \overline{H}, G)$ (see Figure 4.3). Since G and H are not isomorphic, $\mathcal{P}$ will not be self-complementary (see 4.8). For convenience the vertices of G, $\overline{H}$, $\overline{H}$ and G are labelled u_i, v_i, w_i and x_i respectively. We define a function τ' which maps a node labelled

$$\mathcal{P} - [u_1, \ldots, u_j, v_1, \ldots, v_k, w_1, \ldots, w_l, x_1, \ldots, x_m]$$

onto a node labelled

$$\overline{\mathcal{P}} - [\tau([u_1, \ldots, u_j]), \tau^{-1}([v_1, \ldots, v_k]), \tau^{-1}([w_1, \ldots, w_l]), \tau([x_1, \ldots, x_m])].$$

If the u_i's, v_i's, w_i's and x_i's are interspersed, the mapping is defined similarly. It is not difficult to check that $\tau' : T_{\mathcal{P}} \to T_{\overline{\mathcal{P}}}$ is an isomorphism. □

Codes and information

4.43. This section is mainly concerned with Kratochvíl's [229] results on codes over sc-graphs. There have been very few other contributions regarding self-complementary graphs in this area. Lovász [241] has shown that the Shannon zero-error capacity of a vertex-transitive self-complementary graph on n vertices is exactly $\sqrt{n}$; and Marton produced a related probabilistic bound in [245].

4.44. We recall that $d(u, v)$ denotes the distance between two vertices. The closed neighbourhood $N[v]$ consists of v and all its neighbours. A 1-*perfect code* C in a graph G is a set of *code-vertices* whose closed neighbourhoods partition $V(G)$. In other words,

A. no two code-vertices are adjacent, and

B. each vertex not in C is adjacent to exactly one code-vertex.

This implies that for any two vertices $u, v \in C$, $d(u, v) \geq 3$.

The Cartesian square of a graph G is defined by

- $V(G^{[2]}) = V(G) \times V(G)$
- $(u_1, v_1) \sim (u_2, v_2)$ iff $\begin{cases} u_1 = v_1 \text{ and } u_2 \sim v_2 \text{ in } G, \text{ or} \\ u_2 = v_2 \text{ and } u_1 \sim v_1 \text{ in } G. \end{cases}$

The usual way of visualising this is as a square grid of vertices where each row or column induces a graph isomorphic to G. A *permutational code* in $G^{[2]}$ is a code with exactly one code-vertex in each row and column; such a code must have size $|V(G)|$.

Another useful way of defining the edge-set of $G^{[2]}$ is as follows:

$$d((u_1, v_1), (u_2, v_2)) = d(u_1, u_2) + d(v_1, v_2).$$

With these definitions, we now turn to Kratochvíl's results [229, 230].

4.45. Proposition. *If G is a self-complementary graph on n vertices, then $G^{[2]}$ contains a (permutational) 1-perfect code of size n.*

Proof: Let σ be an antimorphism of G, and define $C = \{(u, \sigma(u)) | u \in V(G)\}$. Since σ is a bijection, C will be a permutational 1-perfect code. $\square$

4.46. Proposition. *For any graph G on n vertices, if C is a 1-perfect code in $G^{[2]}$ then $|C| \geq n$.*

Proof: Suppose there is a vertex $u_0 \in V(G)$ such that, for all $v \in V(G)$ $(u_0, v) \notin C$. But any vertex $(u_0, v) \in V(G^{[2]})$ must be adjacent to some code-vertex. So for each $v \in V(G)$ there is a vertex $u_v \in V(G)$ such that $(u_v, v) \in C$, and thus $|C| \geq n$. $\square$

4.47. So 4.45 shows that the Cartesian square of self-complementary graphs achieve the lower bound of 4.46, using a permutational code. 1-perfect codes exist in Cartesian squares of non-self-complementary graphs, e.g. 1-perfect Lee-error correcting codes over $C_{5k}^{[2]}$. But we now show that if G has diameter

2, or if the 1-perfect code is permutational, or attains the lower bound of 4.46, then G must be self-complementary.

4.48. Lemma. *A permutational* 1*-perfect code* C *exists in* $G^{[2]}$ *if and only if* G *is self-complementary.*

Proof: The "if" part follows from Proposition 4.45.

If C is a permutational code in $G^{[2]}$, there is a permutation σ of the vertices of G, such that $C = \{(u, \sigma(u))|u \in V(G)\}$. Take any pair of distinct vertices u, v. If, moreover, C is 1-perfect we have

$$3 \leq d((u, \sigma(u), (v, \sigma(v)) = d(u, v) + d(\sigma(u), \sigma(v))$$

so if $u \sim v$ then $\sigma(u) \not\sim \sigma(v)$. On the other hand, $(u, \sigma(v))$ is not in C and so it must be adjacent to some code-vertex $(z, \sigma(z))$. But then, either $u = z$ and $\sigma(v) \sim \sigma(z)$; or $\sigma(v) = \sigma(z)$, that is $v = z$, and $u \sim z$. So if $u \not\sim v$ then $\sigma(u) \sim \sigma(v)$, and σ is then an antimorphism for G. □

4.49. Lemma. *If a* 1*-perfect code* C *exists in* $G^{[2]}$*, where* $diam(G) = 2$*, then* C *is permutational and* G *is self-complementary.*

Proof: Since $diam(G) = 2$, no row or column of $G^{[2]}$ may contain more than one code-vertex, otherwise we would have two code-vertices at distance at most 2. But by 4.46 $|C| \geq |V(G)|$, so every row and column must contain exactly one code-vertex. C is thus permutational and by 4.48 G is self-complementary. □

4.50. Theorem. *A* 1*-perfect code* C *of size* $|V(G)|$ *exists in* $G^{[2]}$*, if and only if* C *is permutational and* G *is self-complementary.*

Proof: The "if" part follows from Proposition 4.45.

Now let C be a 1-perfect code in $G^{[2]}$ of size $|V(G)|$. If C is permutational the result follows from 4.48.

If C is not permutational then, since it has size $|V(G)|$, there must be some vertex $u_0 \in V(G)$ such that, for all $v \in V(G)$, $(u_0, v) \notin C$ (or some vertex v_0 such that $(u, v_0) \notin C$ for all $u \in V(G)$; the treatment is entirely analogous). But any vertex $(u_0, v) \in V(G^{[2]})$ must be adjacent to some code-vertex. So for each $v \in V(G)$ there is a vertex $u_v \in V(G)$ such that

$(u_v, v) \in C$ and $u_0 \sim u_v$. Since $|C| = |V(G)|$, each u_v must be unique. Denote $A = \{u_v | v \in V(G)\}$; obviously A is not empty, but $u_0 \notin A$.

Now for any $u \notin A$ and $v \in V(G)$ we have $(u, v) \notin C$ (otherwise $u = u_v \in A$), and a code-vertex (x, y) exists such that $(u, v) \sim (x, y)$. Since $(u, y) \notin C$ we have $u \neq x$ and so $v = y$, $x = u_v$ and $u \sim u_v$. As v runs through $V(G)$, u_v runs throughout A, and so for any $u \notin A$ and $w \in A$ we have $u \sim w$. Since $\emptyset \neq A \neq V(G)$, it follows that $diam(G) = 2$ and thus C *is* permutational after all. $\square$

Chapter 5

Generalisations of self-complementary graphs

5.1. The idea of splitting a graph into isomorphic subgraphs is quite natural and there are a number of concepts which are similar in spirit to the self-complementary graphs. Suprunenko [367], for example, constructed a class of 3-uniform sc-hypergraphs. Kocay [224] constructed a similar class, this time vertex-transitive, and proved that a permutation is the antimorphism of some 3-uniform sc-hypergraph if and only if

A. every cycle has even length, or

B. there are either one or two fixed points, and all other cycles have length a multiple of 4.

So 3-uniform sc-hypergraphs on n vertices exist if and only if $n \not\equiv 3 \pmod 4$.

The rest of this chapter is concerned with structural results like this. There is also a number of related enumeration results in Chapter 7, because duality properties of all kinds generally make structures amenable to counting.

Self-complementary and self-converse digraphs

5.2. One of the most natural generalisations of self-complementary graphs is to consider self-complementary digraphs, and in fact they are mentioned several times in other chapters. Another related concept is the self-converse digraphs. The converse D' of a digraph D is obtained by reversing all the arcs of D; if $D' \cong D$, we say that the digraph is self-converse.

We summarise here results given in other parts of the thesis, and present some other theorems on the structure of self-complementary and self-converse digraphs. The self-converse and self-complementary digraphs of order 3 are shown in Figures 5.1 and 5.2, respectively. Note that even on 3 vertices and $\binom{3}{2}$ arcs there are already digraphs that are self-converse but not self-complementary; however, every self-complementary digraph is easily seen to be self-converse.

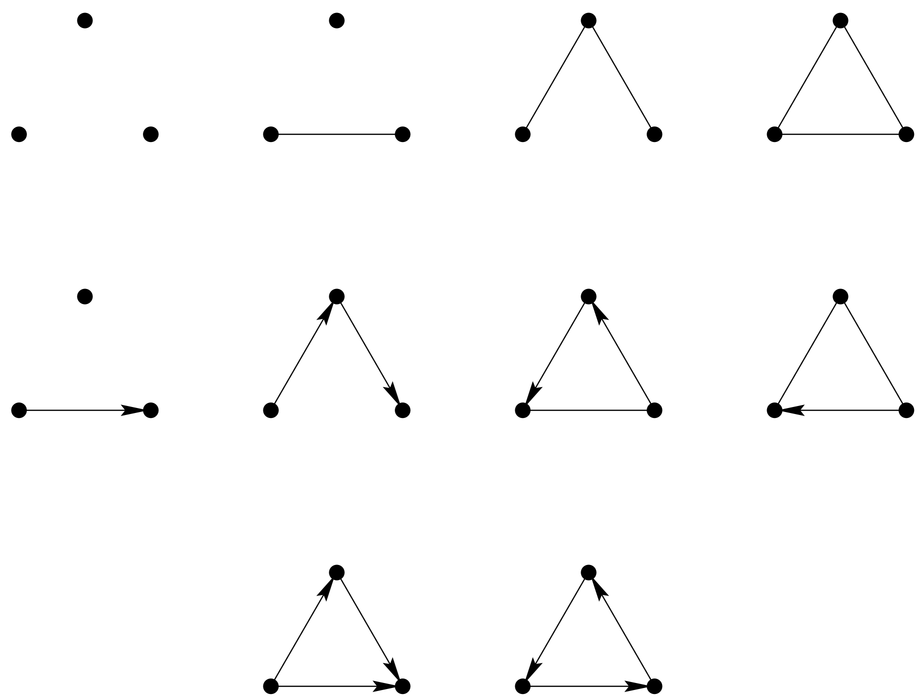


Figure 5.1: The self-converse digraphs of order 3

Every self-complementary graph can also be considered as a self-complementary digraph if we replace each edge by a pair of symmetric (i.e. opposite) arcs; we call these the symmetric self-complementary digraphs. Of course these graphs, when considered as digraphs, are also self-converse, but then every graph is. In particular, the null and complete digraphs are self-converse, and this already hints that the class of self-converse digraphs is rather

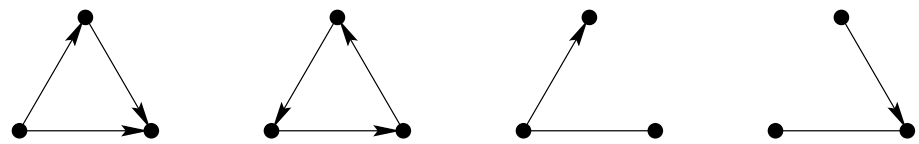


Figure 5.2: The self-complementary digraphs of order 3

too wide to be of interest. It would be interesting to have some results on self-dual digraphs — those which are both self-complementary and self-converse — but so far there are none to speak of; even their enumeration has proved to be too difficult because it involves two dualities (see 7.38).

It is interesting to note, though, that the converse and complement operations commute [174], that is, $\overline{D}' = \overline{D'}$, and so we have the following result [180]:

A. A digraph is self-complementary if and only if its converse is.

B. A digraph is self-converse if and only if its complement is.

However, Cavalieri d'Oro's claim [65] that $D \cong D' \Leftrightarrow L(D) \cong L(D')$ is not true, as can be seen by taking a star with all arcs pointing inward; this is not self-converse, but its line digraph, and the line digraph of its converse are both null.

Since the complement and converse of a tournament are the same thing, the concepts of 'self-complementary tournament', 'self-converse tournament' and 'self-dual tournament' coincide, and we call them just sc-tournaments. They are also the self-complementary oriented graphs, under another name.

The most obvious sc-tournaments are the transitive tournaments, in which vertex i dominates vertex j if and only if $i > j$. These tournaments exist for every integer n, which thus settles the existence question for self-complementary and self-converse digraphs.

5.3. The antimorphisms of all these types of graphs were characterised in 4.12–4.14, where there are also some remarks on self-converse digraphs:

Theorem. *Let π be a permutation of n vertices. Then*

A. π is an antimorphism of some self-converse digraph.

B. π *is an antimorphism of some sc-digraph if and only if it has at most one fixed vertex and all other cycles have even length.*

C. π *is the antimorphism of some sc-tournament if and only if it consists of cycles of even length* $l \not\equiv 0 \pmod 4$*, and at most one fixed vertex.* $\square$

Parts A, B, F, G, H of Theorem 1.29 hold for all these antimorphisms. In particular, if σ is an antimorphism of a self-complementary or self-converse digraph, then for any integer i, σ^i is an antimorphism [automorphism] whenever i is odd [even], and there is a bijection between antimorphisms and automorphisms of the digraph.

The sets of antimorphisms and automorphisms of a digraph are generally disjoint, except when a graph G is considered as a self-converse digraph, in which case the two sets are equal. We note, however, that if G is also a self-complementary digraph, then the set of permutations taking G to its complement is disjoint from the set of those taking G to its converse; this is true of all self-dual digraphs except for sc-tournaments, in which case the two sets are obviously equal. We call both types of permutations antimorphisms, as it is always clear from the context which type we mean.

5.4. What can we say about the order of the antimorphisms and automorphisms of these digraphs. In particular, do they have involutory antimorphisms (that is, antimorphisms of order 2), and non-trivial automorphisms? For sc-graphs the answers are, respectively, 'No' and 'Yes'.

Let us now look at an arbitrary self-converse digraph D. The order of an antimorphism σ of D can be written as $2^s(2t+1)$ for some t. Now, if $s = 0$ then σ^{2t+1} is just the identity permutation, which means that D is its own converse; this happens if and only if D is symmetric, that is, D is essentially a graph.

If D is not symmetric, then $s \geq 1$ and so, by previous results, σ^{2t+1} is a non-trivial antimorphism of order 2^s, while $\sigma^{2(2t+1)}$ is a non-trivial automorphism of D. Further, if $s = 1$ then σ^{2t+1} is an involution, while if $s \geq 2$ then $\sigma^{2^{s-1}(2t+1)}$ is an involutory automorphism.

In both cases ($s = 1$ and $s \geq 2$), the symmetric digraph $D \cup D'$, otained by superimposing D and its converse (possibly giving multiple edges), has $\sigma^{2^{s-1}(2t+1)}$ as a non-trivial involutory automorphism. So the underlying graph of D also has an involutory automorphism.

Now, for self-complementary digraphs we must have $s \geq 1$, so the same arguments hold as above (though of course we are talking about a different kind of antimorphism).

For a sc-tournament s must always be exactly 1, so it must have an involutory antimorphism; it is also evident that no tournament can have an involutory automorphism. We collect these results, all due to Salvi-Zagaglia [343], below:

Theorem. *Let D be a self-complementary or non-symmetric self-converse digraph. Then D has a non-trivial antimorphism whose cycle lengths are all powers of 2, while the underlying graph of D has an involutory automorphism. D itself must have either an involutory antimorphism or an involutory automorphism. If D is a sc-tournament, it must have an involutory antimorphism, but no involutory automorphism.* □

This theorem is "best possible", in the following sense —

- Salvi-Zagaglia showed by an example that there are self-converse digraphs with no involutory antimorphisms, contrary to a claim made by Cavalieri d'Oro [65] (see Figure 5.3)

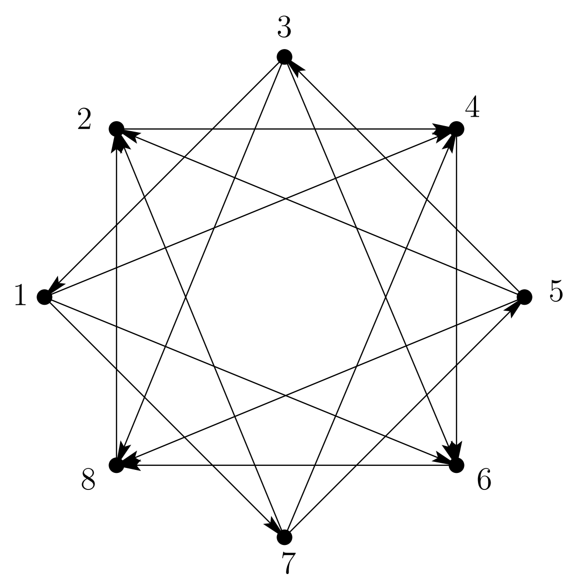


Figure 5.3: A self-converse digraph with no involutory antimorphism

- it is evident that a transitive tournament has trivial automorphism group, because each vertex has a unique out-degree
- Robinson [324] also gave an example of non-transitive sc-tournament with a trivial automorphism group (Figure 5.4).

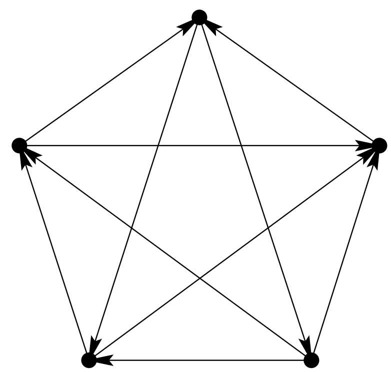


Figure 5.4: A sc-tournament with trivial automorphism group

5.5. In 1.34 we saw that for a sc-graph G the set $F(G)$, of vertices which are fixed by some antimorphism, forms an orbit under the automorphism group of G, a result first proved by Robinson [321]. The proof depends on the structure of antimorphisms of sc-graphs, and so does not carry over to sc-digraphs in general.

Eplett [112] proved a similar result for sc-tournaments using properties of Sylow groups; again, the proof does not extend to sc-digraphs.

However, Robinson [324] showed that his original result *is* valid for sc-digraphs too. He stated his result in the following interesting form — a self-complementary digraph G on $2k+1$ vertices has a unique rooted self-complementary version (up to isomorphism). We give the proof here because it includes the other two results as special cases. As in Chapter 1 we use $u \xrightarrow{\alpha} v$ to mean that u and v are in the same orbit, and α is an automorphism such that $\alpha(u) = v$. We also use O_v and $\mathcal{A}_v$ to denote the orbit and stabiliser of a vertex v, respectively. In the theorem below we deliberately omit any reference to the order of D, because when $|V(D)|$ is even, $F(D)$ will be empty but the results still hold.

5.6. Theorem. *For a self-complementary digraph D, the set $F(D)$ forms an orbit under $\mathcal{A}(D)$. All antimorphisms permute the orbits of D, and $F(D)$ is the unique orbit which is fixed by any (and every) antimorphism.*

Proof: Claim 1: An antimorphism σ maps orbits into orbits. For if u,w, are two vertices in the same orbit, say $\alpha(u) = w$, then $\sigma\alpha(u) = \sigma(w)$. But $\beta := \sigma\alpha\sigma^{-1}$ is an automorphism, and $\beta\sigma(u) = \sigma\alpha(u) = \sigma(w)$, so $\sigma(u)$ and $\sigma(w)$ are also in the same orbit.

Claim 2: For any antimorphism σ and any vertex v, $\sigma O_v = O_v \Leftrightarrow v \in F(D)$. For, if $\sigma(v) = \alpha(v)$ for some automorphism α, then $\alpha^{-1}\sigma$ is an antimorphism, and $\alpha^{-1}\sigma(v) = v$. Conversely, if $\mu(v) = v$ for some antimorphism μ, then $\sigma(v) = \sigma\mu(v)$, which is in the orbit of v since $\sigma\mu$ is an automorphism.

So $F(D)$ is just the union of the orbits which are fixed by any antimorphism; we will now show that it fixes just one orbit. Consider the set of ordered pairs

$$S = \{(v, \sigma) | v \in V(D), \sigma \in \overline{\mathcal{A}}(D), \sigma(v) = v\}.$$

Since each antimorphism has exactly one fixed vertex,

$$|S| = |\overline{\mathcal{A}}(D)| = |\mathcal{A}(D)|.$$

The vertices which appear in some ordered pair of S are just the vertices of $F(D)$, but they may appear in more than one ordered pair. Let x be some vertex of $F(G)$, say $\tau(x) = x$ for some τ. Then the ordered pair $(x, \tau\alpha)$ appears in S if and only if

$$\tau\alpha(x) = x \Leftrightarrow \alpha(x) = \tau^{-1}(x) = x \Leftrightarrow \alpha \in \mathcal{A}_x$$

where $\mathcal{A}_x$ denotes the stabiliser of x. So each vertex $x \in F(G)$ appears in exactly $|\mathcal{A}_x|$ pairs.

Now let z be an arbitrary but definite vertex of $F(D)$. By claim 2, all the vertices of its orbit O_z appear in some ordered pair of S; since the stabilisers of each such vertex have equal size, and each vertex $x \in O_z$ appears in $|\mathcal{A}_x|$ pairs, the vertices of O_z account for $|O_z| \cdot |\mathcal{A}_x|$ pairs, and by the Orbit-Stabiliser theorem they account for all the pairs of S and, thus, all the vertices of $F(G)$. $\square$

5.7. Robinson's theorem allows us to prove the analogue of Rao's theorem [306] for digraphs. As in Chapter 1,

$$N(D) := \{(u, v) \in E(D) | \exists\sigma \in \overline{\mathcal{A}}(D), \sigma(u) = v\}.$$

A regular digraph of degree r is one where every vertex has both indegree and outdegree equal to r.

Theorem. *Let D be a self-complementary digraph. Then the orbits of D can be numbered $V_1, V_2, \ldots, V_{2s}$ if $n = 2k$, or $V_0, V_1, V_2, \ldots, V_{2s}$ if $n = 2k+1$ such that*[1]*:*

A. $|V_0| = 2t + 1$ *for some* t*;*

B. $\sigma(V_0) = V_0$ *and* $\sigma(V_i) = V_{2s+1-i}$ *for any antimorphism* σ*;*

C. $D[V_0]$ *is a regular sc-subgraph (of degree* $(|V_0|-1)/2$*, and* $D[V_i, V_{2s+1-i}]$ *is a regular bipartite self-complementary subgraph (of degree* $|V_i|/2$*) for all* $i \geq 1$*.*

D. $F(D) = V_0$*.*

E. $N(D) = E[V_0] \cup \bigcup_{i=1}^{s} E[V_i, V_{2s+1-i}]$*.*

Proof: Robinson's theorem tells us that $F(D)$ is an orbit, proving D. Now σ maps orbits onto orbits, but only maps $F(D)$ onto itself; and since σ^2 is an automorphism, the mapping on orbits must be an involution; B, C and E then follow (we omit the routine details). This also proves that there are an even number of orbits apart from $F(D)$, and since $|V_i| = |V_{2s+1-i}|$ but $|V(D)|$ is odd, we must have $|F(D)|$ odd, establishing A. □

5.8. It is evident that, for any self-complementary digraph D on $2k+1$ vertices, and for any vertex $v \in F(D)$, $D - v$ is a self-complementary subdigraph which is fixed by some antimorphism of D; we call it a maximal fixed subgraph of D. Actually, we can call it *the* maximal fixed subgraph of D, because the theorem ensures that any two maximal fixed subgraphs of D are isomorphic. However, there can be two non-isomorphic sc-digraphs whose maximal fixed subgraphs are isomorphic.

We note that the representation used by Molina to generate odd order sc-graphs can be extended to sc-tournaments, but not to sc-digraphs in general. If v is a fixed vertex of a sc-digraph D on $2k+1$ vertices, its out-neighbourhood A and in-neighbourhood B must contain k vertices each, but

[1]The results and proofs are stated for $n = 2k+1$. The case $n = 2k$ is analogous and simpler, as any references to V_0 or fixed vertices should just be ignored. See Figure 1.6 for an illustration.

it is only when D is a tournament that A and B are necessarily distinct. Then, if τ is an antimorphism fixing v, we have $\tau(A) = B$ and $\tau(B) = A$, so $B = \overline{A}$. Thus, an odd-order sc-tournament can be represented by a sub-tournament A, and the bipartite sc-tournament C which joins A to B. As with Molina's representation, though, this one does not distinguish between sc-tournaments — there might be two non-isomorphic sc-tournaments with isomorphic representations.

5.9. As in 1.40 we can use the fact that $F(D)$ is an orbit of D to set up some natural bijections. An antisymmetric relation is just a tournament with at most one loop at each vertex. An self-complementary antisymmetric relation must then have $2k$ vertices and exactly k loops; moreover the vertices with and without loops form two exchangeable sets A, B, which we can distinguish unambiguously. So, removing the loops and adding a new vertex v which is dominated by the vertices of A and dominates the vertices of B, we get a unique sc-tournament on $2k+1$ vertices. Conversely, for any sc-tournament T we can take a vertex $v \in F(T)$ and repeat the process, and since the vertices in $F(T)$ are in the same orbit this gives us a unique self-complementary antisymmetric relation on $2k$ vertices. This proves one of the identities of 7.11.

5.10. A digraph is said to be transitive if, whenever there are arcs (a, b) and (b, c), there is also an arc (a, c). There is a unique transitive tournament on n vertices, and it is always self-complementary. For digraphs in general, Hegde, Read and Sridharan [202] showed that the transitive self-complementary digraphs on an even number of vertices can be constructed as follows:

- take the transitive tournament with vertices $\{1, 2, \ldots, 2k\}$, for some k, where i dominates j if and only if $i > j$;
- for each $i, 1 \leq i \leq k$, arbitrarily choose a positive integer n_i;
- choose whether to replace vertex i by a null or complete digraph D_{n_i} on n_i vertices;
- replace vertex $2k + 1 - i$ by $\overline{D}_{n_i}$;
- replace all arcs of the original tournament by bundles of arcs joining every vertex of D_{n_i} to D_{n_j}, for $i \geq j$.

The odd-order transitive sc-digraphs are obtained from the even-order ones by adding a vertex which dominates the $D_{n_1}, \dots, D_{n_k}$, and is dominated by $D_{n_{k+1}}, \dots, D_{n_{2k}}$. So the number of transitive sc-digraphs on $2n$ vertices is the same as the number of such digraphs with $2n+1$ vertices. By the construction algorithm above, it is also equal to

$$\sum f(n_1)f(n_2)\dots f(n_s)$$

where the summation is over all positive integers n_i which sum to n, $f(n_i) = 1$ if $n_i = 1$, and $f(n_i) = 2$ if $n_i > 1$. This gives us the enumeration formulas of 7.36.

5.11. Zelinka [405] defined a curious type of equivalence between digraphs. Two digraphs D_1 and D_2 are said to be isotopic if there are two bijections f, g, both mapping $V(D_1)$ onto $V(D_2)$, such that

$$\forall u, v \in V(D_1), (u, v) \in E(D_1) \Leftrightarrow (f(u), g(v)) \in E(D_2).$$

He then showed that any isotopy between a (possibly infinite) digraph D and its complement must in fact be an isomorphism, that is $f = g$, and so D is self-complementary. However, if we consider the complete digraph with one loop attached to each vertex, then we can only say that there is a digraph isotopic to its complement if and only if at least one of f or g has cycles of only even or infinite length.

5.12. Self-complementary and self-converse digraphs are mentioned in a number of other places in this thesis. For results regarding line-digraphs see 1.55. There are some results on circuits and Hamiltonicity in self-complementary digraphs and tournaments in 2.24–2.30. Chapter 3 is mostly concerned with regular self-complementary digraphs and tournaments. The equivalence of the digraph [tournament] isomorphism problem and the corresponding problem for self-complementary digraphs [tournaments] is proved in 4.5–4.7. See 4.25, 4.27, 4.36–4.39 for results on the subgraphs (self-complementary or otherwise) of a sc-digraph, and the reconstruction conjecture.

For degree sequences, see 6.6, 6.7, 6.8, 6.11, 6.12, 6.23. For enumeration results, see Chapter 7. There is a well-known equality between the number of sc-digraphs on $2k$ vertices, and the number of sc-graphs on $4k$ vertices, but no explicit bijection is known.

Multipartite self-complementary graphs

5.13. Let P be a partition $P = \cup_{i=1}^{r} A_i$ of n vertices, and call the A_i's *classes*. An r-partitioned graph (G, P) is a graph G, such that each edge vw of G has vertices in different classes of P. We say that G is an r-partite graph. An isomorphism of two r-partitioned graphs (G_1, P_1), (G_2, P_2) is just an isomorphism of G_1 and G_2; it need not preserve partitions. The complete multipartite graph with partition P will be denoted by (K, P), while the r-partite complement $(K, P) - (G, P)$ is denoted by $\overline{(G, P)}$. The complement of an r-partite graph is not always unique, as it depends on the partition; that is, we might have $\overline{(G, P_1)} \ncong \overline{(G, P_2)}$ for some partitions P_1, P_2.

In fact [152] the only graphs H for which

$$G_1 \cong G_2 \Rightarrow H - G_1 \cong H - G_2,$$

for any subgraphs $G_1, G_2 \subseteq H$, are $rK_{1,n}, rK_3, K_n, C_5$, and $K_{n,n}$ for some integers r and n.

When we are not particularly interested in a specific partition, but just want to emphasise that the graph has r classes, we denote it by $G(r)$, and its r-partite complement by $\overline{G}(r)$.

An r-partitioned graph (G, P) which is isomorphic to $\overline{(G, P)}$ is called an r-partitioned self-complementary graph, or a graph self-complementary with respect to (K, P). We call G an r-partite self-complementary graph, or r-psc-graph. For bipartite and tripartite self-complementary graphs we use the abbreviations bipsc-graphs and tripsc-graphs, respectively.

5.14. Observations. A sc-graph with respect to $K_{n_1,\dots,n_r}$ has $\frac{1}{2}\sum_{i>j} n_i n_j$ edges, and so we must have $\sum_{i>j} n_i n_j$ even. Thus, for bipsc-graphs at least one of n_1, n_2 must be even, while for tripsc-graphs at least two of n_1, n_2, n_3 must be even. In general, if there are exactly t classes of odd order, this means that $t \equiv 0$ or $1 \pmod 4$. This generalises Lemma 1.2.

When G is uniquely r-colourable, then we do not have to specify the partition, or even the sizes of the classes. In particular, connected bipartite graphs have a unique bipartite complement. Quinn [291] proved a stronger result, that for a given graph G self-complementary with respect to $K_{m,n}$, the factorisation of $K_{m,n}$ into copies of G is unique up to automorphisms of $K_{m,n}$. It is natural to ask whether this is true in general: given two r-partitioned self-complementary graphs $(G_1, P_1), (G_2, P_2)$ if $G_1 \cong G_2$ must P_1 and P_2 also be “isomorphic”? Or could there be, say, a class in P with some distinctive property (e.g. having exactly k vertices, or having two vertices of the same degree s), but no class in Q with the same property?

A self-complementary graph on n vertices is also an n-partite sc-graph and, of course, it is uniquely n-colourable, so this concept includes the usual sc-graphs.

We note that an r-psc graph can be disconnected. For example, $2K_{n,n}$ is a bipsc-graph, and in general $2K_{n_1,\ldots,n_r}$ is self-complementary with respect to $K_{2n_1,\ldots,2n_r}$.

We also note a useful way of constructing successively larger multipartite sc-graphs: given an r-psc-graph G on n vertices, we replace each vertex with a copy of $\overline{K}_k$, and each edge with a copy of $K_{k,k}$. This gives us an r-psc graph on kn vertices, which we call the *k-clone* of G.

5.15. Self-complementary graphs have too many edges to be planar, r-partite or trees (with a finite number of exceptions in each case), as we saw in Chapter 1. Some analogous results are proved in [149]. We note that a cactus is a connected graph whose blocks are either K_2's or cycles.

A. Let (G, P) be an r-partitioned graph, each of whose components is either a tree or a unicyclic graph. Then G is r-psc if and only if it is a bipartitioned graph with one class of size at most 2; one of 20 bipsc-graphs and 14 tripsc-graphs on at most 9 vertices; or one of the small sc-graphs (P_4, C_5, or the A-graph) considered as a 4- or 5-psc-graph.

B. Let G be an r-psc cactus. Then G is one of 22 bipsc-graphs and 20 tripsc-graphs on at most 11 vertices, or the small sc-graphs mentioned in A.

r-partite antimorphisms

5.16. An isomorphism between G and its r-partite complement is called an r-partite antimorphism, or just antimorphism if there is no ambiguity. As usual, an antimorphism is expressed as the product of disjoint cycles:

$$\sigma = \sigma_1 \cdots \sigma_s.$$

A cycle σ_i is said to be *pure* if its vertices are all in the same class; otherwise we say that σ_i is *mixed.* If all the cycles are pure [mixed] then we say that σ is a pure [mixed] antimorphism. We denote the set of antimorphisms of G by $\overline{\mathcal{A}}(G)$, and the set of pure [mixed] antimorphisms by $\overline{\mathcal{A}}_p(G)$ [$\overline{\mathcal{A}}_m(G)$].

When there is no ambiguity, we call these three sets of antimorphisms just $\overline{\mathcal{A}}$, $\overline{\mathcal{A}}_p$ and $\overline{\mathcal{A}}_m$.

For each cycle τ, we define I_τ to be the set of classes of P which intersect σ_τ. So τ is pure if $|I_\tau| = 1$ and mixed if $|I_\tau| > 1$.

5.17. If $\{\sigma_\alpha, \ldots, \sigma_\lambda\}$ is a set of cycles of an antimorphism σ, and if these cycles collectively intersect exactly k classes, then the subgraph induced by their vertices is a k-psc-graph, with antimorphism $\sigma' := \sigma_{alpha} \cdots \sigma_\lambda$.

This implies [149, Thm. 1.6.1] that all the odd length pure cycles of σ must be subsets of the same class; for if any two such cycles were in different classes, they would induce a bipsc-graph with both classes containing an odd number of vertices, which is not possible.

5.18. It is important to note that the results of 1.29 do not necessarily hold for r-partite antimorphisms. To see why, let (G, P) be an r-partitioned graph on n vertices, and colour the edges of K_n blue if they correspond to edges of (G, P), red if they are edges of $\overline{(G, P)}$, and green otherwise. The green edges correspond to the non-edges of (K, P).

An r-partite antimorphism σ must map the blue edges onto red edges, but the red edges may be mapped onto either blue or green edges. Similarly, an automorphism α of G will map blue edges onto blue edges, but red edges need not map onto red edges.

The problem in both cases is that green edges do not always map onto green edges. We therefore define an antimorphism or automorphism of an r-partitioned graph (G, P) to be *periodic* if it maps each A_i into some A_j (possibly $i = j$). In fact, this implies that each A_i is mapped *onto* some A_j:

Lemma. *For any permutation π of the vertices of a complete multipartite graph (K, P), the following are equivalent:*

A. *π permutes the classes of (K, P);*

B. *π is periodic;*

C. *π maps green edges onto green edges.*

Proof: That A $\Rightarrow$ B $\Rightarrow$ C follows from the definitions. Now let π be a permutation mapping green edges onto green edges; then, because there is a finite number of them, π must permute the green edges among themselves. So any two vertices v and w are in different classes if and only if vw is a green edge, if and only if $\pi(v)\pi(w)$ is green, if and only if $\pi(v)$ and $\pi(w)$ are in different classes. Thus C $\Rightarrow$ A. $\square$

So the antimorphisms [automorphisms] which map red edges onto blue [red] edges are precisely the periodic ones. Then parts A, B, F, G, H of Theorem 1.29 hold for periodic antimorphisms and automorphisms. In particular, if σ is a periodic antimorphism then σ^{-1} is also a (periodic) antimorphism, while σ^2 is a (periodic) automorphism; and there is a bijection between periodic antimorphisms and periodic automorphisms of an r-partitioned sc-graph.

The set of periodic antimorphisms [automorphisms] is denoted by $\overline{\mathcal{A}}^*(G)$ [$\mathcal{A}^*(G)$], and obviously $\overline{\mathcal{A}}_p(G) \subseteq \overline{\mathcal{A}}^*(G)$.

5.19. A cycle τ of an antimorphism is said to be k-periodic if it is of the form

$$(u_{11}u_{21}\ldots u_{k1}u_{12}u_{22}\ldots u_{k2}\ldots u_{1\alpha}u_{2\alpha}\ldots u_{k\alpha})$$

where $u_{pq} \in A_{i_p}$ for all p, and $i_p, i_{p'}$ are distinct indices for $p \neq p'$. The cycles of a periodic antimorphism have a nice periodic structure, as we shall see:

Proposition[Gangopadhyay and Rao Hebbare 1982]**.** *Let σ be a periodic antimorphism of an r-partitioned graph (G, P). Let τ be a cycle of σ with $|I_\tau| = k$. Then*

A. the cycle τ is k-periodic;

B. if ψ is any other cycle of σ with $I_\psi \cap I_\tau \neq \phi$, then (1) $I_\psi = I_\tau$ and (2) ψ intersects the classes of $I_\psi = I_\tau$ in the same order as τ;

C. let $|\tau| = k\alpha$, $k \geq 2$; if α is odd, then $k \equiv 0 \pmod 4$.

Proof: Let τ be $(v_1v_2\ldots v_m)$, arranged so that for some t, $v_1, v_2, \ldots, v_t$ is a sequence of vertices in distinct classes $A_{i_1}, A_{i_2}, \ldots, A_{i_t}$, and v_{t+1} is in A_{i_1}. Since σ is periodic we have $\sigma(A_{i_p}) = A_{i_{p+1}}$ for $1 \leq p \leq t-1$, and $\sigma(A_{i_t}) = A_{i_1}$. Since τ is a cycle and $|I_\tau| = k$, we must have $t = k$. Thus τ is k-periodic, and part B follows easily.

For part C we note that τ induces a k-psc-graph, $k \geq 2$, where each class is of size α, so it follows from 5.14 that $k \equiv 0$ or $1 \pmod 4$. But if $k \equiv 1 \pmod 4$, then $k\alpha$ is odd, and $\tau^{k\alpha} = id$ is an antimorphism of this non-trivial induced graph, which is not possible. $\square$

5.20. Corollary. *Let (G, P) be an r-partitioned sc-graph with some periodic antimorphism σ. If τ is a cycle of σ, and $A_i, A_j \in I_\tau$, then $|A_i| = |A_j|$.* $\square$

Bipartite self-complementary graphs

5.21. The bipsc-graphs are the simplest r-psc-graphs, and in fact we know quite a bit about them. Gangopadhyay characterised their degree sequences completely, and Quinn counted the number of small bipsc-graphs; see 6.19 and 7.37, respectively, for these results. These two authors also proved the following, independently:

Proposition[Quinn 1979, Gangopadhyay and Rao Hebbare 1982]**.** *Let G be a connected bipsc-graph with classes A and B. Then*

$$\overline{\mathcal{A}}^*(G) = \overline{\mathcal{A}}(G) = \overline{\mathcal{A}}_p(G) \cup \overline{\mathcal{A}}_m(G).$$

Further, if $\sigma \in \overline{\mathcal{A}}_m(G)$ and τ is a cycle of σ, then the length of τ is a multiple of 4 and τ takes vertices alternately from A and B. $\square$

Proof: Let v be any vertex in A. Then, since G is connected, every other vertex w is at a finite distance from v; moreover, $d(v, w)$ is odd if and only if $w \in B$, and even if and only if $w \in A$. So if an antimorphism maps v from A to B, then all the vertices of A must map to B, and all those of B must map to A, in order to preserve (the parity of) their distances from v. So the cycles of any automorphism must be either all pure or all mixed, and in both cases we get a periodic antimorphism; this proves the first part.

Now, if G is any bipsc-graph (connected or otherwise) with an antimorphism σ mapping A to B and *vice versa*, it is obvious that the vertices of A and B must alternate in each cycle; and that $|A| = |B| = 2k$, for some k, so the number of vertices must be a multiple of 4. Now, any cycle τ of σ induces a bipsc-graph with an antimorphism which interchanges the two classes, so the number of vertices in τ must also be a multiple of 4. $\square$

5.22. The result on cycle lengths was proved by Molina [255] for any bipartite antimorphism which interchanges the two classes — that is, mixed periodic antimorphisms.

We note that a connected bipsc-graph might have only pure antimorphisms — consider the path of length 6, for example, and all its k-clones. It might also have just mixed antimorphisms, as with the graph of Figure 5.5, and all its k-clones.

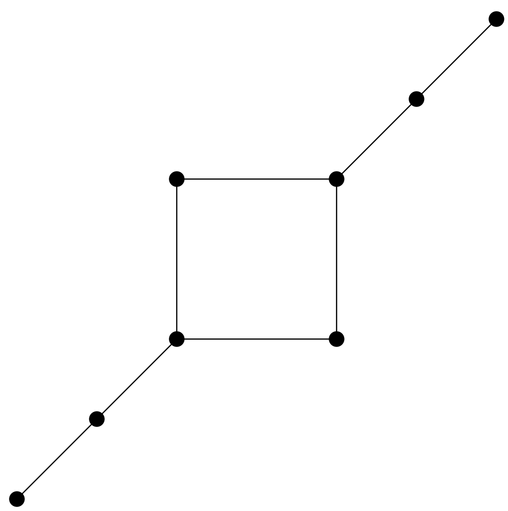


Figure 5.5: A bipsc graph with only mixed antimorphisms

We also note that it follows from Ding [107, Thm. 4.2] that a connected bipsc-graph must contain an induced P_7, J_1 or J_2, where P_7 is the path of length six (see Figure 5.8), and J_1 and J_2 are the graphs in Figure 5.6.

Figure 5.6: J_1 and J_2

5.23. The situation for disconnected bipsc-graphs is different. We cannot split their antimorphisms neatly into pure and mixed — the graph $2K_2$, with classes $\{a, b\}, \{c, d\}$ and edges ac, bd, has an antimorphism $(abd)(c)$. Also, the graph with classes $\{a\}, \{b, c\}$, and edge-set ac has an antimorphism (abc) which is mixed but not periodic. These two graphs (and their k-clones) show that Proposition 5.21 cannot be extended to disconnected bipsc-graphs.

However, we can give a complete characterisation of these graphs. First, we have this result of Gangopadhyay [134]:

Theorem. *The disconnected bipsc-graphs without isolated vertices are precisely the graphs of the form $K_{a,b} \cup K_{a,c}$, for any integers a, b, c.* $\square$

Quinn [291] and Gangopadhyay [134] independently proved that a disconnected bipsc-graph must have a pure antimorphism. This allows us to characterise those disconnected bipsc-graphs not found by Gangopadhyay.

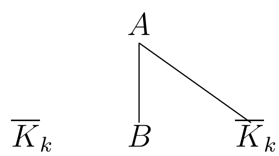


Figure 5.7: Bipsc-graphs with isolated vertices

5.24. Theorem. *The following construction gives all and only bipsc-graphs with isolated vertices (see Figure 5.7):*
Take a bipsc-graph G which has a pure antimorphism σ, and classes A, B (possibly $B = \phi$, in which case G is just a null graph). Then to B add k isolated vertices, and k vertices which are adjacent to every vertex of A.

Proof: It is evident that the construction gives a bipsc-graph G'. For, if we denote the isolated vertices by $v_1, \ldots, v_k$, and the vertices which are adjacent to every vertex of A by $w_1, \ldots, w_k$, then

$$\sigma' := \sigma(v_1 w_1) \ldots (v_k w_k)$$

is a (pure) antimorphism of G'.

Conversely, let (H, P) be a bipartitioned sc-graph with exactly k isolated vertices, and classes A and B. If there is an isolated vertex v, then in $\overline{(H, P)}$

v will be adjacent to every vertex of the other class. So the k isolated vertices must all be in the same class, say B.

By Quinn and Gangopadhyay's result, H has a pure antimorphism π. So B must also contain exactly k vertices of valency $|A|$, and π must map these onto the isolated vertices and *vice versa*. These $2k$ vertices form one or more cycles of π, and the other cycles of π then induce a bipsc-graph with a pure antimorphism, proving the theorem. $\square$

Paths in r-partite sc-graphs

5.25. We know that all sc-graphs have a Hamiltonian path, and they have cycles of all lengths up to $n-2$. There are strong results on path lengths for some classes of bipsc- and r-psc-graphs, as we shall see below, but for example, tripsc-graphs have not been investigated in this regard. We cannot expect any strong results on circuits, at least for bipsc-graphs, as these have no odd-length circuits at all. An interesting question would be to find bounds on the girth and circumference of r-psc-graphs, or some special subclasses such as connected bipsc-graphs.

Theorem[Gangopadhyay and Rao Hebbare 1980a]**.** *Every connected bipsc-graph of order n with $\overline{\mathcal{A}}_m \neq \phi$ has a path of length $n-3$; this result is best possible.* $\square$

5.26. Theorem[Gangopadhyay and Rao Hebbare 1980a]**.** *Let G be a bipsc-graph with antimorphism $\sigma \in \overline{\mathcal{A}}_m \neq \phi$ such that the subgraph induced on each cycle of σ is connected. Then G has a Hamiltonian path.* $\square$

5.27. The next result generalises Clapham and Camion's theorem, mentioned above, on Hamiltonian paths in sc-graphs:

Theorem[Gangopadhyay and Rao Hebbare 1980a]**.** *Let G be an r-psc graph, $r \geq 4$, with an antimorphism σ, such that each cycle of length more than 1 intersects at least four classes. Then G has a Hamiltonian path.* $\square$

Diameters in r-psc-graphs

5.28. One of the strongest, most elegant and best known results on sc-graphs is the one which states that their diameter must be 2 or 3, and their radius exactly 2. This does not hold for r-psc-graphs, if only because they can be disconnected. However, there is a similar theorem for connected r-psc-graphs. We start with bipsc-graphs. The proof given here is original, but a more complicated proof of the bounds on the diameter was given in [148].

Theorem. *Let G be a connected bipsc-graph with diameter d and radius s. Then $3 \leq d \leq 6$ and $3 \leq s \leq 5$; moreover $d = 6 \Rightarrow 3 \leq s \leq 4$.*

Proof: If G is a connected bipsc-graph, with classes A and B, then for any $u \in A$ there must be a vertex $v \in B$ which is not adjacent to u. Moreover, u and v cannot have a common neighbour. So all the vertices of A have eccentricity at least 3, and the same holds for vertices of B, thus establishing the lower bounds.

For the upper bounds we prove a Nordhaus-Gaddum type result that if a connected bipartite graph G has diameter at least 6 then its bipartite complement $\tilde{G}$ has diameter at most 6 and radius at most 4. In particular, if G has radius 6 or more (and, thus, diameter at least 6) $\tilde{G}$ has radius at most 4 and so cannot be isomorphic to G.

Let G be any connected bipartite graph with a vertex u_0 of eccentricity at least 6; that is, there is a vertex w of G whose distance from u_0 is exactly 6. Let A, B be the classes of G, where $u_0 \in A$, and let $u_0, u_1, u_2, u_3, u_4, u_5, u_6 = w$ be a path joining u to w. (See Figure 5.8).

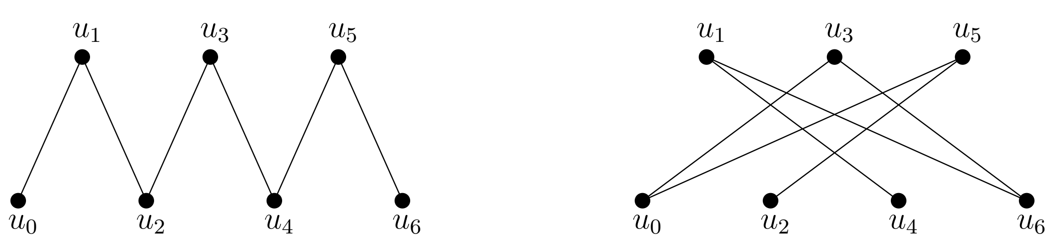


Figure 5.8: A path of length 6 and its bipartite complement

Obviously, the even labeled vertices are in A, the odd labeled vertices in

B. In $\tilde{G}$, $u_0, \ldots, u_6$ induce a path $u_2, u_5, u_0, u_3, u_6, u_1, u_4$. We now consider the distance $\tilde{d}(x, y)$ between two vertices x, y, in $\tilde{G}$.

Case 1: $x, y \in A$. Since no vertex of A can be adjacent to both u_1 and u_5 in G, in $\tilde{G}$ x and y must either have u_1 or u_5 as a common neighbour, or else x (say) is adjacent to u_1 and y to u_5. In either case we have

C1. $\tilde{d}(x, y) \leq 6$, and

C2. $\tilde{d}(x, u_0), \tilde{d}(y, u_0) \leq 4$.

Case 2. $x, y \in B$. No vertex of B can be adjacent to both u_0 and u_6 in G, and so we see that C1 and C2 hold here as well.

Case 3. $x \in A, y \in B$. There are four possibilities to consider in $\tilde{G}$, namely $x \sim u_1, y \sim u_0$, $x \sim u_1, y \sim u_6$, $x \sim u_5, y \sim u_0$ and $x \sim u_5, y \sim u_6$. In each case it can be checked that C1 and C2 are valid. □

We note that if G is a bipsc-graph, a bipartite antimorphism between G and $\tilde{G}$ may very well map one class into another, but this does not affect the proof.

5.29. For r-psc-graphs in general, we have the following:

Theorem[Gangopadhyay and Rao Hebbare 1980b]. *Let G be a connected r-psc-graph with diameter d, where $r \geq 3$. Then $2 \leq d \leq 5$.* □

For every $r \geq 2$ and every diameter d allowed by the previous theorems, Gangopadhyay and Rao Hebbare constructed a single r-psc-graph with diameter d. Their k-clones then provide us with an infinite family of r-psc-graphs with diameter d.

They also proved the following result, which generalises 1.6.A.

5.30. Theorem. *Let G by an r-psc-graph, $r \geq 3$. If G has a periodic antimorphism such that each cycle of length greater than* 1 *intersects at least three classes, then G has diameter* 2 *or* 3.

5.31. Apart from the bounds on diameters of multipartite sc-graphs there has also been much work on extremal problems — questions such as, "What is the smallest n for which there is an r-psc-graph with n vertices and diameter d?"

We say that a complete r-partite graph is *d-decomposable* if it is decomposable into two factors with the same diameter d. If, moreover, the factors are isomorphic (that is, they are r-partite self-complementary), we say that the graph is *d-isodecomposable*. For a complete multipartite graph to be isodecomposable, it must have an even number of edges; that is, if it has s classes of odd order then s must be 0 or 1 $\pmod 4$; we say that such a graph is *admissible*. Here and in the rest of this section, a complete r-partite graph is assumed to be non-trivial, that is, at least one of the classes must contain two or more vertices, so we exclude K_r as an r-partite graph.

We denote by $f_r(d)$ [resp. $g_r(d)$] the smallest number of vertices of a complete r-partite d-decomposable [d-isodecomposable] graph. We denote by $f'_r(d)$ the smallest integer such that for every $n \geq f'_r(d)$ there is an r-partite d-decomposable graph on n vertices. If no such number exists we put $f_r(d) = \infty$, $f'_r(d) = \infty$, $g_r(d) = \infty$ or $g'_r(d) = \infty$.

Finally, we denote by $h_r(d)$ the smallest integer such that every admissible complete r-partite graph with at least $h_r(d)$ vertices is d-isodecomposable; again, we put $h_r(d) = \infty$ if the required integer does not exist.

For all d and all $r \geq 2$, Tomová [371] and Gangopadhyay [136, 138] independently proved that $f_r(d) = f'_r(d)$, while Fronček and Širáň [125, 126, 130] proved that $g_r(d) = g'_r(d)$, and so we have

$$f_r(d) = f'_r(d) \leq g_r(d) = g'_r(d) \leq h_r(d).$$

5.32. For bipartite and tripartite graphs we have not only extremal results, but also characterisations of which graphs are d-decomposable or d-isodecomposable.

Theorem[Tomová 1977, Gangopadhyay 1982a, Fronček 1996a]**.** *A complete bipartite graph $K_{m,n}$ is $d-$ decomposable if and only if it is d-isodecomposable, if and only if at least one of m and n is even and one of the following conditions holds*

A. $d = 3, \quad m \geq 6, \quad n \geq 6;$
B $d = 4, \quad m \geq 4, \quad n \geq 4, \quad$ *or* $m = 3, n \geq 6;$
C. $d = 5, \quad m \geq 3, \quad n \geq 4;$
D. $d = 6, \quad m \geq 3, \quad n \geq 4.$

The extremal results are:

(a) $f_2(3) = \ g_2(3) = 12, \ \ h_2(3) = \infty,$
(b) $f_2(4) = \ g_2(4) = 8, \ \ h_2(4) = \infty,$
(c) $f_2(5) = \ g_2(5) = 7, \ \ h_2(5) = \infty,$
(d) $f_2(6) = \ g_2(6) = 7, \ \ h_2(6) = \infty.\square$

5.33. Theorem[Gangopadhyay 1983, Fronček 1996a]**.** *A complete tripartite graph $K_{m,n,s}$ is d-isodecomposable if and only if at most one of m, n and s is odd and one of the following conditions holds*
A. $d = 2, \ \ m \geq 4, \ \ n \geq 4, \ \ s \geq 5;$
B $d = 3, \ \ m \geq 2, \ \ n \geq 2, \ \ s \geq 2,$ *or* $m = 1, n \geq 4, s \geq 4;$
C. $d = 4, \ \ m \geq 1, \ \ n \geq 2, \ \ s \geq 2;$
D. $d = 5, \ \ m \geq 1, \ \ n \geq 2, \ \ s \geq 4.$

The extremal results are:
(a) $f_3(2) = \ g_3(2) = 13, \qquad h_3(2) = \infty,$
(b) $f_3(3) = \ g_3(3) = 6, \qquad h_3(3) = 8,$
(c) $f_3(4) = \ g_3(4) = h_3(4) = 5,$
(d) $f_3(5) = \ g_3(5) = 7, \qquad h_3(5) = 8.\square$

5.34. Theorem[Gangopadhyay 1983, Fronček 1996b]**.** *For* $r = 4$ *we have*
$f_4(2) = 7, f_4(3) = 5, f_4(4) = 6, f_4(5) = 8,$ *while for* $r \geq 5$,
$f_r(2) = r+1, f_r(3) = r+1, f_r(4) = r+2, f_r(5) = r+4;$
(a) $g_r(2) = g_r(3) = g_r(4) = r+3, g_r(5) = r+5,$ *if* $r \equiv 0 \pmod 4$,
(b) $g_r(2) = g_r(3) = r+1, g_r(4) = r+3, g_r(5) = r+6$ *if* $r \equiv 1 \pmod 4$,
(c) $g_r(2) = g_r(3) = r+1, g_r(4) = r+2, g_r(5) = r+4$ *if* $r \equiv 2 \pmod 4$,
and
(d) $g_r(2) = g_r(3) = g_r(4) = r+2, g_r(5) = r+4$ *if* $r \equiv 3 \pmod 4$. $\square$

Almost self-complementary graphs

5.35. So far we have restricted ourselves to decompositions of complete graphs (or complete digraphs, or complete multipartite graphs) into two isomorphic subgraphs. We will consider in the next section what happens when we have any number of isomorphic subgraphs, but we must first address a limitation of the classical case. Decompositions of K_n into two isomorphic subgraphs are only possible when K_n has an even number of edges, that is $n \equiv 0$ or $1 \pmod 4$. We saw in 2.31 how this limits the use of self-complementary graphs in finding exact values of Ramsey numbers.

To remedy this deficiency we define an almost complete graph on n vertices to be $\tilde{K}_n := K_n - \tilde{e}$, for some edge $\tilde{e}$; and for any graph $G \subseteq \tilde{K}_n$, we denote $\tilde{K}_n - G$ by $\tilde{G}$. This does not give us any ambiguity because $K_n \not\subseteq \tilde{K}_n$, and we will not be referring to $\tilde{K}_n - \tilde{K}_n$ (which in any case is just $\overline{K}_n$). A graph G is then said to be almost self-complementary if

$$G \cong \tilde{G}.$$

Now $\tilde{K}_n$ has an even number of edges exactly when K_n has an odd number of edges, so that almost self-complementary graphs (asc-graphs) can only exist when $n \equiv 2$ or $3 \pmod 4$, and they have $\frac{(n+1)(n-2)}{4}$ edges.

We note that before forming $\tilde{G}$ we have to distinguish the end-vertices v, w, of the *missing edge* $\tilde{e} = vw$; we will call these the *special vertices*. The asc-graphs on 6 vertices are shown in Figure 5.9.

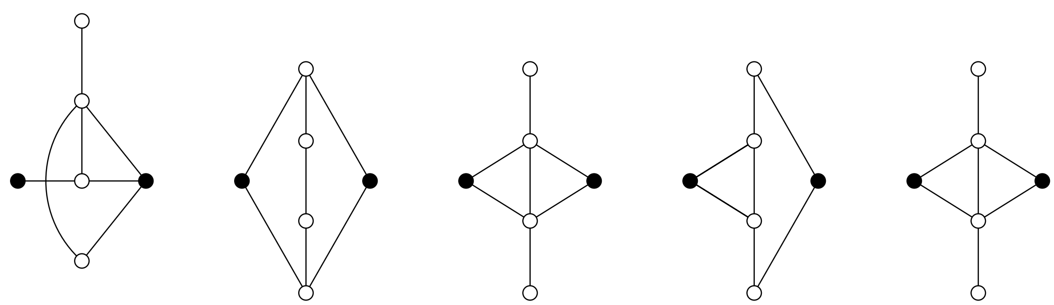


Figure 5.9: The almost self-complementary graphs on 6 vertices; the special vertices are drawn in black.

As usual an isomorphism between G and $\tilde{G}$ is called an antimorphism; if it maps the special vertices onto each other (or keeps them both fixed), it is called a *strong* antimorphism, otherwise we call it a *weak* antimorphism. Strong antimorphisms therefore resemble the periodic r-partite antimorphisms, because their squares are always automorphisms.

5.36. Self-complementary graphs were first studied independently by two authors [Sachs 1962, Ringel 1963] who made many similar discoveries. Three decades later almost the same happened with almost self-complementary graphs — two independent papers were published a year apart [Clapham

1990, P.K. Das 1991] with significant overlap (though each reported results not found by the other). However the first results in this area, namely those on the existence of asc-graphs, were published in 1985 by Chartrand *et al.* [72] in their study of graphs with switching number $n-2$, and this provided part of the inspiration for Clapham's paper.

We recall from 4.31 that if G is almost self-complementary then its self-complement index is $n-1$ (though the converse is not true). So in this sense, the asc-graphs are as close as one can get to being self-complementary on $n \equiv 2$ or $3 \pmod 4$ vertices; but they are not unique in being this close, because for all n there are other graphs on n vertices with $s(G) = n-1$.

It is when we look at the switching number that the asc-graphs really stand out. The switching number of a graph G is n if and only if G is self-complementary, and it is never $n-1$. The switching number of G is $n-2$ if and only if G is almost self-comlementary, so the asc-graphs are actually *the* closest graphs to sc-graphs.

5.37. Our first results characterise the disconnected asc-graphs, and show their existence for all $n \equiv 2$ or $3 \pmod 4$. They are the only results reported, at least in part, by all three authors.

We denote by $x \cup H$ the graph obtained by adding an isolated vertex x to H, and by $y + H$ the graph obtained from H by adding a new vertex y and joining it to every vertex of H.

Proposition[Chartrand *et al.* 1985, Clapham 1990, P.K. Das 1991]**.** *A disconnected graph G is almost self-complementary if and only if $G = x \cup (y + G')$, for some self-complementary graph G'. Moreover, x and y must be the special vertices, and every antimorphism of G is strong, interchanging x and y.*

Proof: It is evident that if G is as above, and we take x and y to be the special vertices of G, then $G \cong \tilde{G}$. We note that G has one isolated vertex and one other component.

Conversely, let G be a disconnected asc-graph on n vertices. If G has no isolated vertex, then every component has at least two vertices; so its complement $\overline{G}$ contains a spanning bipartite graph with at least two vertices in each class. Then $\tilde{G} = \overline{G} - e$ is also connected, so G cannot be almost self-complementary.

So let G have an isolated vertex u. If there is any other isolated vertex v, then $\tilde{G}$ does not contain any isolated vertex, which is a contradiction (there are two cases to check — either u, v are both special vertices, or at least one of them is not).

So u is the only isolated vertex of G. If u is not a special vertex, then in G there must also be a unique vertex v of degree $n-1$, which is not a special vertex either. In $\tilde{G}$, u and v switch roles, but since they are adjacent in exactly one of G, $\tilde{G}$, we get another contradiction.

So u must be a special vertex, and G must contain at least one vertex w of degree $n-2$. If the other special vertex of G has degree less than $n-2$, then $\tilde{G}$ has no isolated vertices, which is impossible. So u and w must be the special vertices of G, and it is then obvious that $G' := G - \{u, w\}$ must be self-complementary.

G' has $n-2$ vertices, and because it is self-complementary each vertex has degree at most $n-4$ in G'; so w must be the unique vertex of degree $n-2$ in G. Thus every antimorphism must interchange u and w. □

5.38. We also note [P.K. Das 1991] that since every self-complementary graph contains a Hamiltonian path, the non-trivial component of G is pancyclic. The proposition tells us that there is a natural bijection between sc-graphs on n vertices and asc-graphs on $n+2$ vertices with an isolated vertex. Since we know that sc-graphs exist for all $n \equiv 0$ or $1 \pmod 4$ we have a neat existence result:

Corollary. *There is a disconnected almost self-complementary graph on n vertices if and only if $n \equiv 2$ or $3 \pmod 4$.* □

So for every natural number n there is a self-complementary or almost self-complementary graph on n vertices.

5.39. The first property of sc-graphs which we studied was their diameter. This parameter has also been extensively studied for r-partite sc-graphs (see the previous section) and t-complementary graphs (see 5.49). For almost self-complementary graphs the corresponding result is short and sweet. The proof we give is adapted from that of P.K. Das, which is much shorter than Clapham's.

Theorem[Clapham 1990, P.K. Das 1991]**.** *If G is a connected almost self-complementary graph, then its diameter and radius are either 2 or 3.*

Proof: Because $\tilde{G}$ is connected, G can have no vertices of degree $n-1$, so the radius (and thus diameter) of G is at least 2.

Now let G be a connected asc-graph of diameter at least 4. Then $\tilde{G}$ also has diameter at least 4, but $\overline{G} = \tilde{G} + e$ has diameter at most 2 (see 1.5.D or [365]). So there are two vertices v, w in $\tilde{G}$ joined by a shortest path v, a, b, c, w of length 4. It can be checked that v, w is the only edge we can add so that every two of these vertices is now at distance at most 2. (If v and w were to be at distance 5 or more, this would be impossible, so we already have $\mathrm{diam}(G) \leq 4$). So v, w, are the special vertices.

Let A be the set of neighbours of v in $\tilde{G}$, C the set of neighbours of w in $\tilde{G}$, and B be $V(G) - A - C - \{v, w\}$. (See Figure 5.10). Because

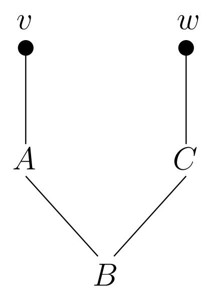


Figure 5.10: Two vertices at distance 4 in $\tilde{G}$.

$4 \leq d(v, w) < \infty$, B must be non-empty, and there can be no edges between A and C. It is then a matter of routine to check that G has diameter at most 3, giving us a contradiction. $\square$

5.40. Theorem[P.K. Das 1991]**.** *For every $n \equiv 2$ or 3 $\pmod 4$, $n \geq 6$, there is a connected asc-graph on n vertices of diameter 2, and one of diameter 3.*

Proof: Let G be any self-complementary graph on $4k+1$ vertices with antimorphism σ whose fixed vertex is v; let A be the neighbourhood of v, and B the set of non-neighbours of v. Then $|A| = 2k = |B|$, $\sigma(A) = B$ and $\sigma(B) = A$. We also recall that for $n \geq 5$, we can specify G to be of diameter either 2 or 3.

If we add a vertex w adjacent to every vertex of A, we get an almost sc-graph G_1 on $4k+2$ vertices, with special vertices v and w; moreover, diam(G_1) = diam(G). If, instead, we join w to every vertex of B, we get an asc-graph G_2 of diameter 3. If, to G_1 and G_2, we add a (non-special) vertex z, which we join to w and to every vertex of A, we get two more asc-graphs on $4k+3$ vertices — G_3, whose diameter is the same as G, and G_4 whose diameter is 3. □

5.41. As with all types of self-complementary structure, we need to know as much as possible about their antimorphisms. Because of the presence of special vertices, the structure of these antimorphisms is unusually complex:

Theorem[Clapham 1990, P.K. Das 1991]**.** *A permutation σ is the antimorphism of an almost self-comlementary graph G on n vertices with special vertices v, w if and only if*

A. $n \equiv 2 \pmod 4$ *and either*

1. *v and w are fixed points of σ, and all other cycles have lengths that are multiples of 4, or*
2. *σ has one cycle of length $4s+2$ containing v and w, with $\sigma^{2s+1}(v) = w$, and all other cycle-lengths are multiples of 4*

B. $n \equiv 3 \pmod 4$, *and either*

1. *σ has a cycle of length 3 containing v and w, and all other cycles have lengths that are multiples of 4, or*
2. *σ has one fixed point, one cycle of length $4s+2$ containing v and w, with $\sigma^{2s+1}(v) = w$, and all other cycle-lengths are multiples of 4.* □

We also note that any set of cycles of such an antimorphism induces an asc-graph if they include both special vertices, and a self-complementary graph otherwise.

5.42. Finally we gather a number of results by Das which parallel similar results on self-complementary graphs. We recall that a biregular graph has exactly two distinct degrees; if these degrees are s, $s+1$, for some s, we say that the graph is almost regular.

A. There are no regular asc-graphs, but for every $k \geq 1$ there is a biregular asc-graph on $4k+2$ vertices, and an almost regular asc-graph on $4k+3$ vertices.

B. Any almost self-complementary graph on $4k + 2$ or $4k + 3$ vertices contains k disjoint induced P_4's.

C. Gibbs' Theorem (1.49) on the $(0, 1, -1)$-adjacency matrix for sc-graphs also holds for almost self-complementary graphs.

5.43. We have generalised sc-graphs to bipartite self-complementary graphs and, in this section, almost self-complementary graphs. Fronček [127] joined the two concepts to form bipartite almost self-complementary graphs. As before, we have $\tilde{K}_{m,n} = K_{m,n} - \tilde{e}$. Actually, we are only interested in the case when m and n are both odd, as otherwise $|E(\tilde{K}_{m,n})|$ is odd, which defeats the whole purpose of the exercise.

We also define the graph $L_{2m,n+1} := K_{2m,n+1} - K_{m,1}$; that is, we remove a star whose end-vertices are in the even part, and whose central vertex is in the odd part of $K_{2m,n+1}$.

Theorem. *A disconnected almost self-complementary factor of $\tilde{K}_{2m+1,2n+1}$ must be one the following*

A. a graph with two components — a star $K_{1,n}$ and the graph $L_{2m,n+1}$

B. a graph with just one non-trivial component, and one or more isolated vertices which are all in the same class. Moreover, if one of the special vertices v, w, is isolated, then it is the only isolated vertex, and every antimorphism interchanges the two classes (so $m = n$) and maps v to w. □

5.44. Theorem. *An almost complete bipartite graph $\tilde{K}_{m,n}$ with m, n odd, $m \leq n$, is decomposable into two isomorphic factors with given diameter d if and only if one of the following conditions holds:*

A.	$d = 3$	$m \geq 7$,		
B.	$d = 4$	$m \geq 3$,	$n \geq 5$,	*or* $m, n \geq 5$,
C.	$d = 5$	$m \geq 3$,	$n \geq 5$,	*or*
D.	$d = 6$	$m \geq 3$,	$n \geq 5$.□	

Isomorphic decompostitions

5.45. Up to now we have looked at decompositions of complete graphs (or digraphs, or multipartite graphs, etc.) into two isomorphic subgraphs. The widest possible way of generalising these concepts is to consider decompositions of any graph H into any number t of isomorphic factors. If such a decomposition exists, we say that t divides H, written $t|H$. If H has a decomposition into t copies of some graph G, then we also write $G|H$, and we denote the set of all such graphs by H/t. We call the decomposition a G-decomposition of H. Of course there might be several G-decompositions of a given graph H; two G-decompositions $(G_1, \ldots, G_t)$, $(G_1', \ldots, G_t')$, are said to be isomorphic if there is an isomorphism of H which maps each G_x to some G_y'.

There has been a tremendous amount of work done on this concept. To give just one example, many papers have been devoted just to the decompositions of K_n into one-factors (see [253] and its references). For information on G-decompositions and various open questions we recommend the series of papers by Harary, Robinson, Wallis and Wormald [187, 188, 189, 192, 193, 194, 195, 323, 392], and the excellent survey book by Bosák [45]. We will give here just a sample of the more important results, restricting ourselves to the case when H is a complete graph.

The most obvious place to start is with the existence question: if H is a graph, and t is an integer, when does t divide H? It is obvious that

$$t|H \Rightarrow t||E(H)|.$$

This is known as the divisibility condition. Which of the integers satisfying the divisibility condition actually divide H? For many types of complete graphs it was established, in a remarkable series of divisibility theorems, that the answer is "all of them".

We use $\mathrm{D}K_n$ to denote the complete digraph on n vertices, and $K_n m$ to denote the complete multipartite (or equiparite) graph with n classes of m vertices each; so $K_n = K_n(1)$.

Divisibility Theorems. *For any integers t and n*

$t\|K_n \Leftrightarrow t\|\frac{n(n-1)}{2}$	*[Harary, Robinson, Wormald 1977/78a, Schönheim and Bialostocki 1978/9]*
$t\|DK_n \Leftrightarrow t\|n(n-1)$	*[Harary, Robinson, Wormald 1978c]*
$t\|K_n(m) \Leftrightarrow t\|m^2\frac{n(n-1)}{2}$	*[Wang 1982, Quinn 1983]*
$t\|DK_n(m) \Leftrightarrow t\|m^2n(n-1)$	*[Wang 1983]*
$t\|K_{a,b,c} \Leftrightarrow t\|ab+bc+ca$ (t even)	*[Yang 1995]* $\square$

Harary, Robinson and Wormald [193] noted that it is obvious that $t|K_{m,n}$ iff $t|mn$; they also showed that, for t odd and $m > t(t+1)$, the divisibility condition is not sufficient for $K_{1,1,m}$, so that Yang's result cannot be improved.

5.46. The self-complementary graphs on n vertices are just the graphs in $K_n/2$. It is therefore natural for the graphs in K_n/t to be called t-complementary graphs, or t-c-graphs for short. The name "t-complementary graph" is taken from Bernaldez, its abbreviation adapted from Gangopadhyay, who used t-sc-graph to refer to what we will call a t-c-class, which is not really a graph.

The results above show that t-complementary graphs on n vertices exist for all feasible t and n (many of Gangopadhyay's results in [139] follow as corollaries). We now consider when a fixed graph G is a t-complementary on n vertices (G itself may have less than n vertices, but we add isolated vertices to make it a factor of K_n, rather than just a subgraph of K_n). The set of all such values n is called the spectrum of G-decompositions of K_n, and obviously G is t-complementary if and only if $t \cdot |E(G)| = \binom{n}{2}$ for some n in the spectrum, so we get a divisibility condition on the number of edges:

$$G|K_n \Rightarrow |E(G)|\,|\frac{n(n-1)}{2}.$$

There is another divisibility condition on the degrees. If $\gcd(G)$ denotes the greatest common divisor of the degrees of G, it can easily be seen that

$$G|K_n \Rightarrow \gcd(G)|\gcd(K_n) = n-1.$$

That these conditions are eventually sufficient is not trivial at all, but was established by Wilson [388]:

Theorem. *Let G be a finite simple graph with m edges. Then there exists an integer $n(G)$ such that, for all $n \geq n(G)$, G divides K_n if and only if*

A. $m|\frac{n(n-1)}{2}$

B. $\gcd(G)|n-1$. $\square$

When $\gcd(G) = 1$ this means that G is t-complementary for all $t \geq \binom{n(G)}{2}/|E(G)|$. Wilson also proved a similar, slightly more complex result for digraphs.

5.47. Wilson's theorem does not give us the value of $n(G)$, nor does it tell us what exceptions there are for $n < n(G)$. So the problem of finding the exact spectrum for a graph G is still interesting. We might also consider the spectrum of G-decompositions of λK_n, the complete multigraph with n vertices, each pair of which are joined by λ edges. Quite a lot is known about small graphs, and certain simple graphs like paths and cycles. In fact, for the small self-complementary graphs the conditions above are always sufficient:

$P_4\|\lambda K_n \Leftrightarrow 3\|\frac{\lambda n(n-1)}{2}$	[Tarsi 1983]
$C_5\|\lambda K_n \Leftrightarrow 5\|\frac{\lambda n(n-1)}{2}, 2\|\lambda(n-1)$	[Rosa and Huang 1975]
$A\|K_n \Leftrightarrow 5\|\frac{n(n-1)}{2}$	[Bermond *et al.* 1980]

5.48. So we know when, for example, P_4 is a t-complementary graph. That P_4 is self-complementary (that is, $t = 2$ is one of the admissible values) is just a co-incidence, but we can use it to impose further conditions on the decomposition. A G-decomposition of H is said to be *complementary* if replacing each copy of G by its complement gives a $\overline{G}$-decomposition of H; we denote this by $G\overline{|}H$. When, moreover, $G \cong \overline{G}$, we say that the decomposition is self-complementary. The self-complementary spectrum for the small sc-graphs has been found:

$P_4\overline{\|}K_n \Leftrightarrow 3\|n-1$	[Granville *et al.* 1989]
$C_5\overline{\|}K_n \Leftrightarrow 5\|\frac{n(n-1)}{2}, 2\|(n-1), n \neq 15$	[Lindner and Stinson 1984]
$A\overline{\|}K_n \Leftrightarrow 5\|\frac{n(n-1)}{2}, 2\|(n-1), n \neq 15$	[Rodger 1992]

Actually Rodger gave the self-complementary spectrum for A-decompositions of λK_n (for which, ironically, we do not have the usual spectrum):

A. If $\lambda \equiv 1, 3, 7$ or $9 \pmod{10}$ then $n \equiv 1$ or $5 \pmod{10}$, and if $\lambda = 1$ then $n \neq 15$.

B. If $\lambda \equiv 2, 4, 6$ or $8 \pmod{10}$ then $n \equiv 0$ or $1 \pmod 5$.

C. If $\lambda \equiv 5 \pmod{10}$ then $n \equiv 1 \pmod 2, n \neq 3$.

D. If $\lambda \equiv 0 \pmod{10}$ then $n \neq \{2, 3, 4\}$.

The self-complementary spectrum for P_4-decompositions of λK_n is given in [109]. If we want a self-complementary P_4-decomposition of K_n containing a smaller self-complementary P_4-decompostion, say, of K_m, then we must have $n \equiv m \equiv 1 \pmod 3$. The only extra condition needed is that $n \geq 3m + 1$, as was shown by Rees and Rodger [317]. Rees and Stinson proved this for $n \geq 616$ in [318].

Self-complementary decompositions are also used in [24].

5.49. As in 5.31 we can investigate extremal problems on diameters using the following definitions. Let $F_m(d)$ [resp. $G_m(d)$] be the smallest number of vertices of a complete graph that can be decomposed into m factors [isomorphic factors] of diameter d. We denote by $H_m(d)$ the smallest integer such that every admissible complete graph with at least $H_m(d)$ vertices can be decomposed into m isomorphic factors of diameter d; here, we say that K_n is admissible if $m|\binom{n}{2}$. If the requested numbers do not exist we put $F_m(d) = \infty$, $G_m(d) = \infty$ or $H_m(d) = \infty$, and obviously we have

$$F_m(d) \leq G_m(d) \leq H_m(d).$$

Thus, the results of Chapter 1 (1.5–1.6) on diameters can be phrased as

$$F_2(d) = G_2(d) = H_2(d) = \begin{cases} 4, & \text{if } d = 3 \\ 5, & \text{if } d = 2 \\ \infty, & \text{otherwise.} \end{cases}$$

5.50. For the case when $m = 3$, Kotzig and Rosa [228] showed that

A. $F_3(d) = G_3(d) = H_3(d)$ for $d = \infty$ and $d = 1, 3, 4, 5, 6$, and $G_3(\infty) = 3$, $G_3(1) = \infty$, $G_3(3) = G_3(4) = G_3(5) = 6$, $G_3(6) = 9$.

B. $12 \leq F_3(2) \leq G_3(2) = H_3(2) \leq 13$ (but see below).

C. $H_3(d) \leq 3d - 6$.

They also showed that when $m = p^r$, p an odd prime,

$$G_m(\infty) = H_m(\infty) = m,$$

and conjectured that $G_m(d) = H_m(d)$ for all $m \geq 2$, $d \geq 2$.

Híc and Palumbíny [207] proved that for any $m \geq 3$, $H_m(2) \leq 6m$ and for any $m \geq 46$, $G_m(2) = H_m(2) = 6m$. It is noted in a footnote that a result of R. Nedela implies that the last equality holds for $m \geq 22$; and by Kotzig and Rosa's result, it also holds for $m = 3$.

Cyclically t-complementary graphs

5.51. If G is a t-complementary graph, then a (G, t)-complementary class (or just t-complementary class) is a particular G-decomposition $(G_1, \ldots, G_t)$ of K_n[2]. In the classical case of self-complementary graphs ($t = 2$) every 2-complementary graph immediately defines a 2-complementary class. This does not happen for $t \geq 3$; in fact we might get non-isomorphic (G, t)-complementary classes. Two (G, t)-complementary classes, $(G_1, \ldots, G_t)$ and $(G'_1, \ldots, G'_t)$, are isomorphic if there is a permutation of the n vertices mapping each G_x onto some G'_y.

If $(G_1, \ldots, G_t)$ is a t-complementary class, and σ_i is a permutation mapping G_i onto G_{i+1} (subscripts taken modulo t), then we call the ordered sequence $(\sigma_1, \ldots, \sigma_t)$ a t-morphism class. In the particular case when $\sigma_1 = \sigma_2 = \ldots = \sigma_t := \sigma$ we have a cyclic t-morphism σ; we call such a class a cyclic (G, t)-class, while G is said to be a cyclically t-complementary graph, after Bernaldez. The cyclic (G, t)-class is determined fully by G and σ, being just $(G, \sigma(G), \sigma^2(G), \ldots, \sigma^t(G))$, so we can say that σ is a cyclic t-morphism of G.

We note that Gangopadhyay, who did a lot of work in this area, called t-complementary classes "t-sc-graphs", t-morphism classes "complementing permutation classes", and cyclic t-morphisms "stable complementing permutations".

5.52. If $C = (G, \sigma G, \ldots, \sigma^t G)$ is a cyclic t-c-class with cyclic t-morphism σ, then for any permutation α, $\alpha C = (\alpha G, \alpha\sigma G, \ldots, \alpha\sigma^t G)$ is a cyclic t-c-class with cyclic t-morphism $\alpha\sigma\alpha^{-1}$. If α does not commute with σ then we get two different cyclic morphisms; it would be interesting to know if the reverse can happen, that is having two different, non-isomorphic (G, t)-classes with the same cyclic t-morphism.

[2] We will use (G, t)-class and t-c-class as abbreviations.

Problems. *A. Can there be two non-isomorphic cyclic (G,t)-classes? B. Can they have the same cyclic t-morphism?*

Another interesting line of enquiry would be to investigate those graphs G whose (G,t)-classes are all cyclic. One would suspect that these are exceedingly rare; and that there are cyclically t-c-graphs whose non-cyclic (G,t)-classes far outnumber the cyclic ones.

Problems. *C. Construct an infinite class of cyclically t-c-graphs G_i, such that the proportion of (G_i,t)-classes which are cyclic tends to 0 as $i \to \infty$. D. Is there a graph G which is cyclically t-complementary for an infinite number of t's, but where the proportion of (G,t)-classes which are cyclic tends to 0 as $t \to \infty$?*

5.53. The structure of cyclic t-morphisms can be found by a straightforward adaptation of the result by Sachs and Ringel for $t = 2$ (see 4.12). The method of proof was essentially first used for t by Guidotti in [165]. Schönheim (unpublished) used it in order to help Harary *et al.* [192] obtain an even more general result (the Divisibility Theorem for K_n), though they later got to know about Guidotti's proof. However, these authors all stated their result as "There exists a t-complementary graph on n vertices whenever $t | \binom{n}{2}$ and either $\gcd(t,n) = 1$ or $\gcd(t,n-1) = 1$." Our corollary is stronger than this.

Surprisingly, although more than a decade later Bernaldez [40, 41] and Gangophadhyay [139, 140] both investigated the structure and existence of cyclic t-morphisms at length, and although both came close to this result and its corollary, neither of them stated the results formally. Apparently, different terminology prevented them from seeing the link between their work and that of the other authors.

5.54. Theorem. *Let t be an odd [resp. even] integer. A permutation is a cyclic t-morphism if and only if it has cycle lengths a multiple of t [resp. $2t$] except, possibly, for one fixed point.*

Proof: We first prove necessity. Let $\tau = (v_1, v_2, \ldots, v_k)$ be a cycle of a cyclic t-morphism σ. Then the edge v_1v_2 generates a cycle of length k (unless $k = 2$, in which case the edge cycle has length 1). So we must have $k \equiv 0 \pmod{t}$. If $t = 2p$ is even, and $k = 2p(2q+1)$ is an odd multiple of t, then the edge $v_1 v_{p(2q+1)+1}$ generates a cycle of length $p(2q+1)$, which is not a multiple of t. So for t even, the cycle lengths must be a multiple of $2t$.

If σ has two fixed points (v) and (w) would have a vertex-pair cycle vw of length 1 which is impossible. So σ can have at most one fixed point.

To prove the converse, we note that a permutation σ of n vertices induces a permutation of the edges of K_n. If the edge cycles thus induced all have length a multiple of t, then we can colour the edges in each cycle with colours $1, 2, \ldots, t$ in that order, thus defining a cyclic t-class. (In fact, all cyclic t-classes with cyclic t-morphism σ will be produced in this way).

Given a permutation with cycle lengths satisfying the conditions of the theorem, we thus want to show that for any edge $e = vw$ of K_n, we only have $\sigma^x(e) = e$ when x is a multiple of t. Now e can be mapped onto itself in just three ways:

Case1: v and w are in the same cycle τ of σ, and $\sigma^x(v) = v$, $\sigma^x(w) = w$. If τ has length k, this will happen exactly when x is a multiple of k, and thus of t.

Case 2: v and w are in the same cycle τ of σ, and $\sigma^x(v) = w$, $\sigma^x(w) = v$. Then $\tau = (v_1, v_2, \ldots, v_{2s})$, $v = v_i$, $w = v_{s+i}$, and this case arises whenever $x = 2rs + s$ is an odd multiple of s. When t is odd, $t|2s \Rightarrow t|s$. When t is even, $2t|2s \Rightarrow t|s$. In either case, x is a multiple of t.

Case 3: v and w are in different cycles of σ, and $\sigma^x(v) = v$, $\sigma^x(w) = w$. Then at least one of the cycles must have length a multiple of t, and so x must be a multiple of t. $\square$

5.55. Corollary. *A cyclically t-complementary graph on n vertices exists if and only if*

A. $n \equiv 0$ *or* $1 \pmod t$, *for t odd,*

B. $n \equiv 0$ *or* $1 \pmod{2t}$, *for t even,*

that is, if and only if t divides either n or $n-1$, and $t|\binom{n}{2}$. $\square$

5.56. Of course, this result is just a special case of the Divisibility Theorem. Any integer t dividing $\frac{n(n-1)}{2}$ will divide K_n, but if it does not divide n or $n-1$ then all t-complementary graphs must be non-cyclic, which supports the Harary-Robinson [189] conjecture that almost all t-complementary graphs on n vertices are non-cyclic as $n \to \infty$. We note that Schwenk counted the number of non-isomorphic cyclically t-complementary graphs on n vertices (see 7.42–7.44).

The construction used in proving the Divisibility Theorem for K_n gives us a way of finding non-cyclically t-complementary graphs — we just apply it for some t which does not divide n or $n-1$. Meanwhile, Alavi, Malde, Schwenk and Swart [17] constructed, for each k, a graph G_k on $9k+10$ vertices which is 9-complementary but has no cyclic 9-morphism. So we also have examples of non-cyclically t-complementary graphs when t *does* divide $n-1$.

5.57. Corollary 5.55 can be used to show that Thm. 4.1 of [139] is wrong. Gangopadhyay claimed that any cyclically t-complementary graph on n vertices can be extended to a cyclically t-complementary graph on $n+t$ vertices. For t even, this is obviously wrong, so let us see where the fault is. The construction is as follows: let $(G_1,\ldots,G_t)$ be a cyclic t-class with vertices $v_1,\ldots,v_{kt}$ for some k. Add vertices $w_1,w_2,\ldots,w_t$, and to each G_i add the edges w_iv_j, $1\le j\le kt$, and w_iw_{i+1}. For $t=2$ the construction is not well-defined, as w_1w_2 is simultaneously assigned to G_1 and G_2. For $t>3$ the construction gives a cyclic decomposition of a graph $H\ne K_{n+t}$. It is only for $t=3$ that the construction works.

Theorem 4.2 of the same paper fails for similar reasons.

5.58. Given a decomposition $(G_1,\ldots,G_t)$ of K_n, it is obvious that if some permutation σ maps each G_i to G_{i+1} for $1\le i\le t-1$, then it also maps G_t to G_1, so σ is a cyclic t-morphism. Even if σ maps G_i to G_{i+1} for $1\le i\le t-2$, and σ^s is the identity, for some $s\not\equiv t-1$, then σ must map G_{t-1} to G_t, and must thus be cyclic [140].

To show that this result cannot really be improved Gangopadhyay constructed, for every odd t, a non-cyclic decomposition of K_t into t almost one-factors; and for every even t, a non-cyclic decomposition of K_{2t} into t double stars. In each case, there is a t-morphism permuting the first $t-1$ subgraphs cyclically, while leaving the last subgraph fixed.

Here an almost one-factor is a one-factor with an isolated vertex; this graph also has a cyclic t-morphism [329]. The double star on $2k$ vertices consists of two stars on k vertices, with an edge joining the two vertices of degree $k-1$ in each star.

5.59. There are a number of properties of cyclic t-morphisms which we give here without proof:

Theorem[Bernaldez 1994, 1996, Gangopadhyay 1994]**.** *Let σ be a cyclic t-morphism of a cyclically t-complementary graph G on n vertices. Then σ^s is an automorphism of G if and only if $s \equiv 0 \pmod{t}$, and a cyclic t-morphism of G if and only if s and t are relatively prime.*

The sum of the degrees (in G) of any t successive vertices in a cycle is $n-1$, while the degree of the fixed vertex, if any, is $\frac{n-1}{t}$. If G is regular, all vertices have degree $\frac{n-1}{t}$ (even if there is no fixed vertex).

Any set of cycles of σ induces a cyclically t-complementary subgraph G. Any cyclically t-complementary graph on st vertices can be extended to one on $st+1$ vertices. □

"Super-symmetrical" graphs

5.60. Self-complementary graphs are partitions of the edge-set of K_n into two isomorphic subgraphs. We can also consider partitions of the vertex-set of a graph G such that the subgraphs induced by the two subsets, A, B, are isomorphic. However, this is not such an interesting concept in itself; for one thing, it ignores completely the edges between A and B; for another, every graph is isomorphic to the subgraphs in some bisection, so that the definition is too wide. Kelly and Merriel considered, instead, the much more restrictive class of bisectable graphs (and, later, digraphs). A graph G is *bisectable* if it has $2n$ vertices, and for *each* set S of n vertices, the subgraphs induced by S and $V(G) - S$ are isomorphic.

We denote by $G \times K_2$ the graph consisting of two copies of G joined by a one-factor; by $\vec{K_n}$ the transitive tournament of order n; and by DG the digraph obtained from a graph G by replacing each edge by a pair of opposite arcs.

Theorem[Kelly and Merriel 1960]**.** *A graph is bisectable if and only if it is one of the following graphs or it complement: $2C_4$, K_{2n}, $2K_n$, nK_2 and $K_n \times K_2$,.* □

It is interesting to note that none of these graphs are self-complementary, while in the next theorem the only self-complementary digraphs are $\vec{K_{2n}}$, and two tournaments of order 4 and 6, respectively; however, they are all self-converse, with the exception of one tournament of order 4.

5.61. Theorem[Kelly and Merriel 1968]**.** *A digraph D is bisectable if and only if D or $\overline{D}$ is one of the following:*

A. DG, for any graph G in the previous theorem;

B. $n\vec{K}_2$, $2\vec{K}_n$, $\vec{K}_{2n}$;

C. seven exceptional digraphs of order 4, six of order 6 and one of order 8. □

5.62. Harary, meanwhile, took an interest in the graphs which, no matter how they are oriented, always give a self-converse digraph. For convenience, the complete graphs on one and two vertices are denoted by C_1 and C_2.

Theorem[Harary, Palmer and Smith 1967]**.** *The only connected simple graphs for which every orientation is self-converse are the "small cycles" C_1, C_2, C_3, C_4 and C_5. The only connected multigraphs which are always self-converse are just the small cycles, the multigraphs whose underlying graph is C_2, and the multigraphs obtained from all these graphs by adding the same number of loops at each vertex.* □

5.63. Instead of orienting a graph, we can assign a positive or negative sign to all the vertices, edges, or both vertices and edges; the resulting graph is called a marked graph, signed graph or net, respectively. In each case, the dual is obtained by reversing all the signs, and self-dual graphs are defined in the obvious way.

A graph which, no matter how its vertices or edges (or both) are signed, always produces a self-dual marked graph, signed graph or net, is called *M-dual*, *S-dual* or *N-dual*, respectively. It is not difficult to see that an M-dual graph is bisectable and that, conversely, all the bisectable graphs found by Kelly and Merriel in 1960 are M-dual. The S-dual and N-dual graphs are given below.

Theorem[Harary and Kommel 1979]**.** *The only S-dual graphs without isolated vertices are C_6, $2C_3$, $2C_4$, $2K_{1,n}$, $2nK_2$, $K_{1,2n}$, $nK_{1,2}$ and $K_{2,n}$.*
The only N-dual graph are $2K_2$ and $\overline{K}_{2n}$. □

Chapter 6

Degree Sequences

6.1. The *degree sequence* of a graph is the sequence of its degrees, conventionally, arranged in non-increasing order. One of the reasons for studying the degree sequences of a class of graphs is to try to find some easy, or relatively fast way of distinguishing members of the class from each other and from non-members. In the case of sc-graphs these problems are just as difficult as the general graph isomorphism problem (see 4.2–4.10), but there are a number of structural properties which can be deduced from the degree sequence alone — for a self-complementary graph G, there are straightforward formulas for calculating the number of triangles and P_4's, and (with the easily handled exception of $G^*(4k)$) for deciding whether G is Hamiltonian or has a 2-factor (see 2.5, 2.6, 2.16 and 2.18). Since it is an easy matter (algorithmically) to detect whether a graph is isomorphic to $G^*(4k)$, these four problems have $O(n)$ or $O(n^2)$ solutions. Knowing more about the degree sequences of sc-graphs opens the prospect of solving other problems too.

Parthasarathy and Sridharan [285] counted self-complementary graphs and digraphs according to their degree sequence, while Hegde and Sridharan [203] did the same for self-converse digraphs and oriented graphs, and sc-tournaments. Their results are useful when generating self-complementary graphs systematically, because as soon as one knows that the required number of non-isomorphic sc-graphs with a given degree sequence has been generated, these graphs can be excluded from consideration; see 4.11.

Self-complementary graphs and digraphs

6.2. Let $\pi = (d_1, d_2, \ldots, d_n)$ be a finite sequence of non-negative integers such that $d_1 \geq d_2 \geq \cdots \geq d_n$ and $d_1 + d_2 + \cdots + d_n$ is even[1]. The sequence π is said to be *graphical* if it is the degree sequence of some graph G (without loops or multiple edges), and we call G a *realisation of* π. Erdős and Gallai [115] showed that a necessary and sufficient condition for π to be graphical is that, for $r = 1, \ldots, n-1$, we have

$$\sum_{i=1}^{r} d_i \leq r(r-1) + \sum_{j=r+1}^{n} \min(r, d_j).$$

If π is the degree sequence of some self-complementary graph, we say that it is *potentially self-complementary.* If, moreover, every realisation of π is self-complementary, we say that it is *forcibly self-complementary.* We note that an antimorphism of order 4 has all cycles of length exactly 4, except for a single fixed vertex whenever n is odd. We therefore call this a 4-*cycle antimorphism*, and define a 4-*cycle realisation* of π to be a self-complementary realisation of π which has an antimorphism of order four. 2-*cycle realisations* are defined similarly for self-complementary or self-converse digraphs.

We saw in 1.21 that the degree sequence of a sc-graph is symmetrical about $\frac{1}{2}(n-1)$, and that for any $r \neq \frac{1}{2}(n-1)$ there is an even number of vertices with degree r. We thus say that π is *suitable* if
either $n = 4k$, for some k, and

A. $d_i + d_{4k+1-i} = 4k - 1$, for $i = 1, 2, \ldots, 2k$

B. $d_{2j} = d_{2j-1}$, for $j = 1, 2, \ldots, 2k$

or $n = 4k + 1$, for some k, and

C. $d_i + d_{4k+2-i} = 4k$, for $i = 1, 2, \ldots, 2k + 1$

D. $d_{2j} = d_{2j-1}$, for $j = 1, 2, \ldots, 2k$.

A suitable sequence $\pi = (d_1, d_2, \ldots, d_n)$ is highly redundant. If $n = 4k$, then we can define a corresponding *reduced sequence* $\pi^* := (a_1, a_2, \ldots, a_k)$

[1] Some people, like the author, are more used to visualising sequences in non-decreasing order; here, instead, the indices increase as the values decrease.

where $a_i = d_{2i-1} = d_{2i}$; we can recover the full suitable sequence from π^* since $\pi = (a_1, a_1, a_2, a_2, \ldots, a_k, a_k, 4k-1-a_k, 4k-1-a_k, \ldots, 4k-1-a_2, 4k-1-a_2, 4k-1-a_1, 4k-1-a_1)$. If $n = 4k+1$, since we know that $d_{2k+1} = 2k$, we can define a similar reduced sequence that represents π uniquely.

6.3. Clapham and Kleitman [90] showed by construction that in fact every graphical suitable sequence is potentially self-complementary. In the first paper on sc-graphs [341] Sachs had proved the particular case in which the sequence has just one or two distinct degrees, by constructing appropriate regular and biregular sc-graphs on $4k+1$ and $4k$ vertices respectively. Clapham [86] showed that, given a suitable sequence, instead of the Erdős–Gallai conditions we can use a much simpler set of inequalities to check whether the sequence is graphical. We thus have the following:

Theorem [Clapham 1976b]**.** *The sequence π is potentially self-complementary if and only if π is suitable and the corresponding reduced sequence π^* satisfies the inequalities*

$$\sum_{i=1}^{r} a_i \leq r(n-1-r) \text{ for } r = 1, 2, \ldots, k$$

where $n = 4k$ or $4k+1$. Furthermore, if π is potentially self-complementary then it has a 4-cycle realisation. □

6.4. Rao [302] tabulated $\overline{p}_n$, the number of suitable sequences, for small values of n (see Table 6.1), but the general counting formula is still unknown.

n	4	5	8	9	12	13	16	17
$\overline{p}_n$	1	2	4	8	20	39	111	210

Table 6.1: The number of potentially self-complementary degree sequences

6.5. In the case of digraphs we must specify the outdegree and indegree of each vertex, so we now have a *degree-pair sequence*, where each element in the degree sequence is an ordered pair of integers $(d^+(v_i), d^-(v_i))$, where $d^+(v_i)$ is the outdegree of v_i and $d^-(v_i)$ is the indegree of v_i. An *integer-pair sequence* is a sequence of ordered pairs of non-negative integers $(a_1, b_1), \ldots, (a_n, b_n)$. We call it *digraphic* if it is a degree-pair sequence of some digraph or, if it is realised by at least one self-complementary [self-converse] digraph, *potentially digraphic self-complementary* [*potentially self-converse*].

We consider all integer-pair sequences to be in dictionary order, that is $a_i \geq a_{i+1}$, and if $a_i = a_{i+1}$ then $b_i \geq b_{i+1}$. By $\pi_1 = \pi_2$ we mean that if π_1 and π_2 are both arranged in dictionary order the corresponding terms are equal. For any integer-pair sequence $\pi = (a_1, b_1), \ldots, (a_n, b_n)$ we define

$$\overline{\pi} := (n-1-a_1, n-1-b_1), \ldots, (n-1-a_n, n-1-b_n)$$

$$\tilde{\pi} := (b_1, a_1), \ldots, (b_n, a_n).$$

Digraphic integer-pair sequences were characterised by Fulkerson [132], but a more useful formulation can be found in Chen [73, p.405]. The potentially digraphic self-complementary, and potentially self-converse sequences were characterised by Das [103], solving a problem posed by Rao [302]. A second problem he posed, the characterisation of forcibly digraphic self-complementary sequences is still open.

6.6. Theorem [Das 1981]**.** *The integer pair sequence π is potentially digraphic self-complementary if and only if the following conditions hold:*

A. π is digraphic

B. $\pi = \overline{\pi}$

C. whenever n is even, say $n = 2k$, we have $\sum_{i=1}^{k}(a_i + b_i) \equiv k \pmod 2$.

Every potentially digraphic self-complementary sequence has a 2-cycle realisation. □

6.7. Theorem [Das 1981]**.** *The integer pair sequence π is potentially self-converse if and only if π is digraphic and $\pi = \tilde{\pi}$. Every potentially self-converse sequence has a 2-cycle realisation.* □

6.8. The following result was found by Eplett [112]; it also follows from the proof of 6.7.

Theorem. *[Das 1981] The integer pair sequence π has a sc-tournament realisation if and only if π is digraphic and $\tilde{\pi} = \pi = \overline{\pi}$; equivalently [Eplett 1979] if and only if, for $1 \le i \le \frac{1}{2}n$, we have*

$$a_i + b_i = n - 1 = a_i + a_{n+1-i}, \text{ and}$$
$$\sum_{j=1}^{i} b_j \ge \binom{i}{2}.$$

Moreover, every such sequence has a 2-cycle tournament realisation. □

See also Sridharan and Merajuddin [360] for the number of degree sequences of sc-tournaments.

Edge degree sequences

6.9. For a graph [digraph] the degree of an edge uv is the ordered pair $(d(u), d(v))$ $[(d^-(u), d^-(v))]$ of degrees [indegrees] of its vertices, and the *edge-degree sequence* is the sequence of degrees of its edges; this concept was introduced in [172]. An edge-degree sequence with at least one self-complementary graph [digraph] realisation will be said to be *potentially edge [digraphic] self-complementary*. If, moreover, all of its realisations are self-complementary we say that it is *forcibly edge self-complementary*.

For any integer-pair sequence $\pi = (a_1, b_1), \ldots, (a_m, b_m)$ we define the sequences $A := (a_1, \ldots, a_m)$, $B := (b_1, \ldots, b_m)$, $C = (a_1, b_1, \ldots, a_m, b_m)$; and the set $S := \{d_1, \ldots, d_r\}$ consisting of all distinct integers appearing in π, labelled so that $d_1 > d_2 > \cdots > d_r$. We also define the following parameters:

$k'(s,t)$ is the number of times the ordered integer pair (s,t) occurs in π,

$k(s,t) := \begin{cases} k'(s,t) + k'(t,s) & \text{if } s \neq t \\ k'(s,t) & \text{if } s = t \end{cases}$

$k(s)$ is the number of times s occurs in A

$k'(s)$ is the number of times s occurs in B

$n(s)$ is the number of times s occurs in C

$l'_i := \begin{cases} k'(d_i)/d_i & \text{if } d_i \neq 0 \\ n(d_i) & \text{if } d_i = 0 \end{cases}$

If π is the edge-degree sequence of a graph, then it will contain only positive integers, and the number of vertices of degree d_i will be just $n(d_i)/d_i$.

So for sequences consisting only of positive integers (*positive integer-pair sequences*) we define

$$l_i := n(d_i)/d_i, \text{ for } 1 \le i \le r.$$

With this notation we can now state the results obtained by Das [104], some of which were also found by Chernyak [74, 75].

6.10. Theorem [Das 1983, Chernyak 1983a, 1983b]**.** *A positive integer-pair sequence π is potentially edge self-complementary if and only if the following conditions hold:*

A. $l_i = l_{r+1-i}$ *and* l_i *is an even integer, for* $1 \le i \le \frac{1}{2}r$.

B. *If* r *is odd, then* $l_{(r+1)/2} \equiv 1 \pmod 4$.

C. $k(d_i, d_i) + k(d_{r+1-i}, d_{r+1-i}) = \frac{1}{2} l_i(l_i - 1)$ *for* $1 \le i \le \frac{1}{2}(r+1)$.

D. $k(d_i, d_{r+1-i}) = \frac{1}{2} l_i^2$ *for* $1 \le i \le \frac{1}{2}r$.

E. $k(d_i, d_j) + k(d_{r+1-i}, d_{r+1-j}) = k(d_i, d_{r+1-j}) + k(d_{r+1-i}, d_j) = l_i l_j$ *for* $i \ne j$, $1 \le i, j \le \frac{1}{2}(r+1)$.

F. $k(d_i, d_j)$ *is even for* $i \ne j$, $1 \le i, j \le r$.

Moreover, every potentially edge self-complementary sequence has a 4-cycle realisation. □

6.11. Both in 6.10 and 6.13, Das lists another condition but does not use it in the proof: $d_i + d_{r+1-i} = \sum_{j=1}^{r} l_j - 1$. In fact it can be derived from A, C, D, E by writing $d_i = n(d_i)/l_i = (\sum_{j=1}^{r} k(d_i, d_j) + k(d_i, d_i))/l_i$. It is interesting to note that in 6.10 and 6.13, as well as the next theorem, it is not assumed that π is graphic or digraphic.

Theorem [Das 1983]**.** *An integer-pair sequence π is potentially edge digraphic self-complementary if and only if the following conditions hold:*

A. *For all* i, $1 \le i \le \frac{1}{2}(r+1)$

$$d_i + d_{r+1-i} = \begin{cases} 2\left(\sum_{i=1}^{r/2} l'_i\right) - 1 & \text{if } r \text{ is even,} \\ 2\left(\sum_{i=1}^{(r-1)/2} l'_i\right) + l'_{(r+1)/2} - 1 & \text{if } r \text{ is odd.} \end{cases}$$

B. $k'(d_i) = l'_i d_i$, $k'(d_{r+1-i}) = l'_i d_{r+1-i}$ *and* l'_i *is an integer for* $1 \le i \le \frac{1}{2}(r+1)$.

C. *If* r *is odd, then* $l'_{(r+1)/2} \equiv 1 \pmod 2$.

D. $k'(d_i, d_i) + k'(d_{r+1-i}, d_{r+1-i}) = l'_i(l'_i - 1)$ *for* $1 \le i \le \frac{1}{2}(r+1)$.

E. $k'(d_i, d_j) + k'(d_{r+1-i}, d_{r+1-j}) = l'_i l'_j$ *for* $1 \le i \ne j \le r$.

Furthermore, every potentially edge digraphic self-complementary sequence has a 2-cycle realisation. □

6.12. Theorem [Das 1983]**.** *An integer-pair sequence* π *is the edge-degree sequence of some sc-tournament (and, in particular, of some 2-cycle tournament realisation) if and only if* π *is potentially edge digraphic self-complementary and is the edge-degree sequence of some tournament.* □

6.13. Theorem [Das 1983, Chernyak 1983b]**.** *A positive integer-pair sequence* π *is forcibly edge self-complementary if and only if the following conditions hold:*

A. $l_i = l_{r+1-i} = 2$ *or* 4 *for* $1 \le i \le \frac{1}{2}r$.

B. *If* r *is odd, then for* $i = \frac{1}{2}(r+1)$ *either* $l_i = k(d_i, d_i) = 5$ *or* $l_i = k(d_i, d_i) + 1 = 1$

C. $\{k(d_i, d_i), k(d_{r+1-i}, d_{r+1-i})\} = \{0, \frac{1}{2}l_i(l_i - 1)\}$ *for* $1 \le i \le \frac{1}{2}r$.

D. $k(d_i, d_{r+1-i}) = \frac{1}{2}l_i^2$ *for* $1 \le i \le \frac{1}{2}r$.

E. $\{k(d_i, d_j), k(d_{r+1-i}, d_{r+1-j})\} = \{k(d_i, d_{r+1-j}), k(d_{r+1-i}, d_j)\} = \{0, l_i l_j\}$ *for* $i \ne j$, $1 \le i, j \le \frac{1}{2}(r+1)$. □

6.14. These results can be used for the characterisation of forcibly self-complementary degree sequences, by reducing it to an integer-pair sequence problem. Let $\pi = (d_1, \ldots, d_n)$ be a graphic degree sequence. The integer-pair sequence $S(\pi)$ is defined by the following algorithm based on the one of [171, 221].

Step 1. Put $S(\pi) = \phi$ (the empty sequence) and $V = (d_1, d'_1), \ldots, (d_n, d'_n)$ where $d'_i = d_i$ for $1 \le i \le n$. Go to Step 2.

Step 2. Order V so that the d'_i sequence is non-increasing. Remove the first member of V, say (d_k, d'_k), from V. Let the ith member of V be (d_j, d'_j); for all i, $1 \leq i \leq d'_k$, add (d_k, d_j) to $S(\pi)$ and put $d'_1 = d'_j - 1$. Proceed to Step 3.

Step 3. If for any member of V d'_i is zero, then remove it from V. If $V = \phi$ stop. Otherwise go to Step 2.

Theorem [Das 1983, Chernyak 1983b]. *Let π be a graphic degree sequence. Then π is forcibly self-complementary if and only if $S(\pi)$ satisfies conditions* A *through* E *of 6.13 and condition* F *given below, where δ_{ij} is the Kronecker delta.*

F. If i, j, s, t are such that $i \neq s$, $j \neq t$ and $\delta_{it}\delta_{js} < \min\{k(d_i, d_j), k(d_s, d_t)\}$, then either

$$k(d_i, d_t) = \frac{l_i(l_t - \delta_{it})}{1 + \delta_{it}} \text{ or } k(d_j, d_s) = \frac{l_j(l_s - \delta_{js})}{1 + \delta_{js}}. \square$$

6.15. Rao [307] obtained a different characterisation of the forcibly self-complementary degree sequences, and counted them too. He first wrote π in the form

$$\pi = (d_1)^{n_1} \cdots (d_m)^{n_m}$$

where $\sum n_i = n$, $n_i > 0$, $d_1 > \ldots > d_m > 0$, and $(d_i)^{n_i}$ means that d_i occurs exactly n_i times in π. The characterisation can then be stated as follows.

Theorem. *A degree sequence $\pi = (d_1)^{n_1} \cdots (d_m)^{n_m}$ with $n \equiv 0 \pmod 4$ is forcibly self-complementary if and only if m is even, say $m = 2k$, and for all i, $1 \leq i \leq k$, the following hold:*

A. $n_i = 2$ *or* 4.

B. $n_i = n_{m+1-i}$.

C. $d_i + d_{m+1-i} = n - 1$.

D. $d_i = n - 1 - \frac{1}{2}n_i - \sum_{j=1} i - 1 n_j$, *with the convention that* $\sum_{j=1}^{0} n_j = 0$. $\square$

6.16. Theorem. *A degree sequence* $\pi = (d_1)^{n_1} \cdots (d_m)^{n_m}$ *with* $n \equiv 1 \pmod 4$ *is forcibly self-complementary if and only if* m *is odd, say* $m = 2k+1$, *and the following hold:*

A. $n_{k+1} = 1$ *or* 5.

B. $d_{k+1} = \frac{1}{2}(n-1)$.

C. $\pi' := (d'_1)^{n'_1} \cdots (d'_{m-1})^{n'_{m-1}}$ *is a forcibly self-complementary sequence of length* $n - n_{k+1}$, *where*

$$d'_i := \begin{cases} d_i - n_{k+1}, & 1 \le i \le k, \\ d_i - 1, & k+1 \le i \le 2k, \end{cases}$$

$$n'_i := \begin{cases} n_i, & 1 \le i \le k, \\ n_{i+1}, & k+1 \le i \le 2k. \end{cases} \square$$

6.17. Theorem. *Let* $\overline{f}_n$ *be the number of forcibly self-complementary degree sequences on* n *vertices. Then* $\overline{f}_{4k} = \overline{f}_{4k+1}$ *and* $(\overline{f}_{4k})_{k\ge 0}$ *is the Fibonacci sequence, that is* $\overline{f}_0 = \overline{f}_4 = 1$ *and* $\overline{f}_{4k} = \overline{f}_{4k-4} + \overline{f}_{4k-8}$.

Bipartite sc-graphs and sc-tournaments

6.18. Let (G,P) be a bipartitioned graph, where G is a bipartite graph, P is the bipartition $A = \{u_1, \dots, u_m\} \cup B = \{v_1, \dots, v_n\}$, and where

$$d(u_1) \ge \cdots \ge d(u_m) \text{ and } d(v_1) \ge \cdots \ge d(v_n).$$

If we denote $d(u_i)$ by d_i and $d(v_j)$ by e_j, then the degree sequence of (G,P) is the bipartitioned sequence

$$\pi((G,P)) = (d_1, \dots, d_m | e_1, \dots, e_n)$$

The bipartite complement of (G,P) is $\overline{(G,P)} := K_{|A|,|B|} - G$. We have to specify the bipartition explicitly because disconnected bipartite graphs do not necessarily have a unique bipartite complement. If there is an isomorphism $\sigma : (G,P) \to \overline{(G,P)}$ (called a *bipartite antimorphism*) we say that G is a *bipartite self-complementary* (bipsc) graph. We call σ a pure bipartite antimorphism if it keeps A and B fixed, and a mixed periodic bipartite antimorphism if it interchanges A and B. Every bipsc-graph must have either a pure or a mixed periodic antimorphism; see 5.21–5.23.

A bipartitioned sequence will be said to be *graphic* [*unigraphic*] if it is the degree sequence of at least one bipartite graph [exactly one bipartite graph, up to isomorphism]. A bipartitioned sequence is *potentially bipsc* [*forcibly bipsc*] if it is graphic and its realisations include at least one bipsc-graph [only bipsc-graphs].

We say that $\pi = (d_1, \ldots, d_m | e_1, \ldots, e_n)$ is *evenly balanced* if

$m = 2t = n$ is even,

$d_i + e_{2t+1-i} = 2t$ for $1 \leq i \leq 2t$, and

$d_{2i-1} = d_{2i}$ for $1 \leq i \leq t$;

and *bi-symmetrical* if

$d_i + d_{m+1-i} = n$ for $1 \leq i \leq m$,

$e_j + e_{n+1-j} = m$ for $1 \leq j \leq n$.

We use r^k to denote a sequence $r, \ldots, r$ of length k

6.19. Theorem [Gangopadhyay 1982b]**.** *A bipartitioned sequence $\pi = (d_1, \ldots, d_m | e_1, \ldots, e_n)$ is potentially bipsc if and only if it is graphic and satisfies at least one of the following conditions:*

A. π is evenly balanced.

B. π is bi-symmetrical and exactly one of m and n is odd.

C. π is bi-symmetrical, m and n are both even, and either $d_{m/2} = d_{m/2+1} = \frac{1}{2}n$, or $e_{n/2} = e_{n/2+1} = \frac{1}{2}m$.

D. π is bi-symmetrical, and m, n and $\sum_{j=1}^{n/2} e_j - \sum_{i=1}^{m/2} d_i - \frac{1}{4}mn$ are all even.

Furthermore

(1) π is the degree sequence of some bipsc-graph (G, P) with a mixed periodic bipartite antimorphism iff A *holds, and*

(2) π is the degree sequence of some bipsc-graph (G, P) with a pure bipartite antimorphism iff at least one of B, C, D *holds.* □

6.20. Proposition [Gangopadhyay 1982b]**.** *A bipartitioned sequence* $\pi = (d_1, \ldots, d_m | e_1, \ldots, e_n)$ *is the degree sequence of a connected bipsc-graph iff* π *is potentially bipsc and*

A. $\min(d_m, e_n) > 0$.

B. $\pi \notin \{(n_1, n - n_1 | 1^n), (1^m | m_1, m - m_1)\}$ *for some integers* m_1, n_1, *with* $0 < m_1 < m$ *and* $0 < n_1 < n$. ☐

6.21. For the characterisation of forcibly bipsc sequences we need the following conditions, which we can make without loss of generality since, if

$$\pi = (d_1, \ldots, d_m | e_1, \ldots, e_n)$$

violates any one of them, then

$$\overleftrightarrow{\pi} := (e_1, \ldots, e_n | d_1, \ldots, d_m)$$

satisfies them all.

X. If $d_1 > d_m$ then $e_1 > e_n$.

Y. If some $e_j = \frac{1}{2}m$, then some $d_i = \frac{1}{2}n$.

Z. If $d_1 > d_m$, $e_1 > e_n$, some $d_i = \frac{1}{2}n$ and some $e_j = \frac{1}{2}m$, then $d_p - n + q \geq e_q - m + p$, where $p = \max\{i | d_i > \frac{1}{2}n\}$ and $q = \max\{j | e_j > \frac{1}{2}m\}$.

We denote $\frac{1}{2}m$ by s iff m is even, and $\frac{1}{2}n$ by t iff n is even. If π is evenly balanced we denote $\frac{1}{2}m = \frac{1}{2}n$ by t. Note that if π is bi-symmetrical and $d_i = d_{m+1-i}$ for some i [$e_j = e_n + 1 - i$ for some j] then n [resp. m] is even and thus t [resp. s] is well-defined.

6.22. Theorem [Gangopadhyay 1981]**.** *Let* $\pi = (d_1, \ldots, d_m | e_1, \ldots, e_n)$ *be a bipartitioned sequence satisfying (without loss of generality) conditions X, Y and Z. Then* π *is forcibly bipsc iff* $\sum_{i=1}^m d_i = \sum_{j=1}^n e_j$ *and* π *satisfies one of the following four conditions:*

A. π *is evenly balanced and the sequence* $\pi' := (d_1 + 2r - 1, \ldots, d_{2r} + 2r - 1, e_1, \ldots, e_{2r})$ *is forcibly self-complementary.*

B. π *is bi-symmetrical,* $d_1 = d_m$*,* $e_1 = e_n$ *and either* $\min(s,t) \leq 2$*, or* $\min(s,t) = 3$ *and* $\max(s,t) \leq 4$*.*

C. π *is bi-symmetrical,* $d_1 = d_m$ *and if* k *is the number of* e_j*'s in* π *which are equal to* 0*, then either* $t-k \leq 2$*, or* $\dot{\pi} := ((t-k)^m|e_{k+1}, \ldots, e_{2t-k})$ *is one of the following bipartitioned sequences:*
$\pi_1 = (3^6|3^6)$, $\pi_2 = (4^6|3^8)$, $\pi_3 = (3^8|4^6)$, $\pi_4 = ((t-k)^2|1^{2(t-k)})$,
$\pi_5 = ((t-k)^4|2^{2(t-k)})$, $\pi_6 = ((t-k)^m|(m-1)^{t-k}, 1^{t-k})$,
$\pi_7 = ((t-k)^4|3, 2^{2(t-k-1}, 1)$, $\pi_8 = (3^{2s}|2s-1, s^4, 1)$.

D. π *is bi-symmetric,* n *is even, and if* p *is the number of* d_i*'s greater than* $\frac{1}{2}n$ *and* q *the number of* e_j*'s greater than* $\frac{1}{2}m$*, then* $0 < p \leq \frac{1}{2}m$ *and* $0 < q \leq \frac{1}{2}n$*. Furthermore, if* h *is the number of* e_j*'s in* π *which are not less than* $m-p$*, then*

(1) $\sum_{i=1}^{p} d_i = (n-h)p + \sum_{j=n-h+1}^{n} e_j$

(2) $\sum_{j=1}^{h} e_j = (m-p)h + \sum_{i=m-p+1}^{m} d_i$

(3) *either* $p = \frac{1}{2}m$ *or* $t-h \leq 2$ *or* $\ddot{\pi} := ((t-h)^{m-2p}|e_{h+1}-p, \ldots, e_{2t-h}-p)$ *is one of* π_1 *to* π_8 *, with* t *replaced by* $t-h$ *and* k *replaced by* 0*,*

(4) *the bipartitioned sequence* $\pi^* := (d_1-n+h, \ldots, d_p-n+h|e_{n-h+1}, \ldots, e_n)$ *is unigraphic.* □

6.23. For a bipartite tournament T (i.e. an orientation of $K_{m,n}$), with bipartition $A = \{v_1, \ldots, v_m\} \cup B = \{u_1, \ldots, u_n\}$, where $d^+(u_1) \geq \cdots \geq d^+(u_m)$ and $d^+(v_1) \geq \cdots \geq d^+(v_n)$, we define the degree sequence to be

$$\pi(T) = (a_1, \ldots a_m|b_1, \ldots, b_n),$$

where a_i and b_j denote $d^+(v_i)$ and $d^+(u_j)$ respectively. The indegrees are determined unambiguously by $d^+(v_i)+d^-(v_i) = n$ and $d^+(u_j)+d^-(u_j) = m$. When $m = 2r$ and $n = 2s$ are even, we define

$$\sigma(\pi) := (a_1 + \cdots + a_r) + (b_1 + \cdots + b_s) - rs.$$

The bipartition is unique because T must be connected. T is a bipsc-tournament if it is isomorphic to its bipartite complement. We recall that a 2-cycle bipartite antimorphism must have at most one fixed vertex, all other cycles being of length 2; while a pure bipartite antimorphism must keep A and B fixed. A pure bipartite antimorphism cannot have fixed vertices in both A and B, but it might have, say, two or more fixed vertices in A. Bagga and Beineke announced their characterisation of the degree sequences of bipsc-tournaments with pure, and pure 2-cycle bipartite antimorphisms in [30]:

Theorem. *The bipartitioned sequence $\pi = (a_1, \dots a_m | b_1, \dots, b_n)$ is the degree sequence of some bipsc-tournament with a pure bipartite antimorphism if and only if*

A. π is the degree sequence of some bipartite tournament, that is $\sum_{i=1}^{k} a_i \leq \sum_{j=1}^{n} \min(k, m - b_j)$ for all k, with equality when $k = m$.

B. $a_i + a_{m+1-i} = n$ and $b_j + b_{n+1-j} = m$ for all i, j.

C. m or n is even.

D. Whenever $m = 2r$ and $n = 2s$ are both even, and $a_r > a_{r+1}$ and $b_s > b_{s+1}$, then $\sigma(\pi)$ is also even.

To characterize degree sequences which have a bipsc-tournament realisation with a pure 2-cycle bipartite antimorphism, we replace D with the stricter condition:

D'. Whenever m and n are both even, $\sigma(\pi)$ is also even.

Open problems

6.24. Despite the impressive results listed here, there is still great scope for characterising different types of degree sequences for different types of self-complementary graphs. The most notably absent are the forcibly digraphic self-complementary [302] and forcibly self-converse degree-pair sequences, and the potentially edge self-converse sequences. A less difficult task might be to find which digraphic degree-pair sequences have only sc-tournaments as realisations.

Alternatively, one can turn to more exotic problems. For example Pirzada [288] studied self-converse score lists of oriented graphs; Chartrand, Gavlas, Harary and Schultz [69] and Yan, Lih, Kuo and Change [397] characterised signed degree sequences of signed graphs, which raises the obvious problem of finding the signed degree sequences of self-dual signed graphs. Finally Nair [263] proposes the characterisation of triangle sequences, where the *triangle number* of a vertex v is the number of triangles containing v, and the *triangle sequence* of a graph is the sequence of the triangle numbers of its vertices. Similar definitions can be made for edge-triangle sequences.

Chapter 7

Enumeration

7.1. Enumeration of graphs is a wide and active field of research, much of it triggered by Harary and Palmer, and their classical book [182]. The enumeration of many types of self-complementary structures has also received a great deal of attention, starting with Read's fundamental result on the number of self-complementary graphs and digraphs [313]. Various other self-dual structures have also been enumerated, and Robinson's survey [324] is still a useful reference point. The history of the methods used is a long and controversial one, as they have been attributed to Cauchy and Frobenius, claimed by Burnside and developed by Redfield. The latter only published two papers [315, 316] which went unnoticed, and his results were re-discovered and expanded by Pólya [289], de Bruijn (e.g. [51, 52, 53]), and Harary [173], working also with Palmer [179].

The aim of this chapter is quite different from Robinson's article, because it is intended as a useful collection of results concerning self-complementary graphs and digraphs. In most cases we do not give proofs or explain the methods used, but we do present explicit formulas as far as possible. We also treat topics such as colour cyclic factorisations, sc-k-plexes and sc-orbits, which generalise the enumeration of sc-graphs. We start with some basic definitions.

Definitions

7.2. A partition of n is denoted by the vector $\mathbf{j} = (j_1, j_2, \dots, j_n)$ where j_k is the number of parts equal to k. That is

$$\sum_{k=1}^{n} k j_k = n.$$

A permutation α which has j_k cycles of length k will be said to have type $\mathbf{j}$. The number $p(\mathbf{j})$ of permutations of type $\mathbf{j}$ in S_n is then

$$p(\mathbf{j}) = \frac{n!}{\prod_{k=1}^{n} k^{j_k} j_k!}.$$

We denote the greatest common divisor and least common multiple of two numbers by $\langle r, t\rangle$ and $[r, t]$ respectively.

7.3. Definition.A generating function (or counting polynomial) $g(x)$ for a sequence $\{u_n\}$ is the power series

$$u_0 + u_1 x + u_2 x^2 + \cdots + u_n x^n + \cdots$$

This definition may be extended to sequences with several parameters. For example, the generating function $g(x, y)$ for a sequence $\{u_{n,r}\}$ is

$$u_{0,0} + u_{0,1} y + u_{1,0} x + u_{1,1} xy + \cdots + u_{n,r} x^n y^r + \cdots$$

7.4. Definition. Let Γ be a permuation group acting on an object set $S = \{1, 2, \dots, n\}$. For each permutation $\gamma \in \Gamma$, let $j_k(\gamma)$ denote the number of cycles of length k in the disjoint cycle decomposition of γ. Then the cycle index of Γ is the polynomial

$$Z(\Gamma) = Z(\Gamma; x_1, x_2, \dots, x_n) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} {x_1}^{j_1(\gamma)} {x_2}^{j_2(\gamma)} \dots {x_n}^{j_n(\gamma)}.$$

For any polynomial $p(x)$ we let $Z(\Gamma, p(x))$ denote the polynomial obtained by replacing each x_k by $p(x^k)$. For example

$$Z(\Gamma, 1 + x) = Z(\Gamma; 1 + x, 1 + x^2, \dots, 1 + x^n).$$

7.5. Definition. We denote the symmetric group on n elements by S_n. The permutations in S_n induce permutations of the $\binom{n}{2}$ 2-subsets in the object set. These permutations of the 2-subsets form the pair group $S_n^{(2)}$. Similarly, the permutations of the $n(n-1)$ ordered pairs of distinct elements in the object set form the reduced ordered pair group $S_n^{[2]}$. The permutations induced by S_n on all ordered pairs, where the elements need not be distinct, form the ordered pair group S_n^2. Note that, for $n > 2$, $S_n \cong S_n^{(2)} \cong S_n^{[2]} \cong S_n^2$.

The restricted power group $S_n^{S_2 *}$ consists of ordered pairs $(\alpha;\beta)$ of permutations $\alpha \in S_2$, $\beta \in S_n$ so that for any ordered pair of distinct elements (i,j),

$$(\alpha;\beta)(i,j) = \begin{cases} (\beta i, \beta j) \text{ if } \alpha = (1)(2) \\ (\beta j, \beta i) \text{ if } \alpha = (12) \end{cases}$$

The power group $S_n^{S_2}$ is defined similarly, except that now i and j need not be distinct.

C_n denotes a regular cyclic permutation group of order n and degree n; in other words, it is generated by an n-cycle. It is isomorphic to Z_n, as a permutation group, acting on Z_n as object set.

7.6. The cycle index of the pair groups [278] is given by

$$\begin{aligned} Z(S_n^{(2)}) &= \frac{1}{n!}\sum_{\mathbf{j}} p(\mathbf{j}) \left[\prod_{k=1}^{\lfloor n/2 \rfloor} (x_k x_{2k}^{k-1})^{j_{2k}} {x_k}^{k\binom{j_k}{2}} \right. \\ & \qquad \left. \prod_{k=0}^{\lfloor (n-1)/2 \rfloor} x_{2k+1}^{kj_{2k+1}} \prod_{1 \le r < t \le n} x_{[r,t]}^{\langle r,t \rangle j_r j_t} \right] \\ Z(S_n^{[2]}) &= \frac{1}{n!}\sum_{\mathbf{j}} p(\mathbf{j}) \prod_{k=1}^{n} {x_k}^{(k-1)j_k + 2k\binom{j_k}{2}} \prod_{1 \le r < t \le n} x_{[r,t]}^{2\langle r,t \rangle j_r j_t} \end{aligned}$$

The cycle index of C_n [222] is given by

$$Z(C_n) = \frac{1}{n}\sum_{k=1}^{n} x_{[k,n]/k}^{\langle k,n \rangle} = \frac{1}{n}\sum_{r|n} \phi(r) x_r^{n/r}$$

since the Euler function $\phi(r)$ gives the number of integers $k \le n$ with $\langle k,n \rangle = n/r$.

Self-complementary graphs and digraphs

7.7. We start with a brief overview of Read's method [313] for counting self- complementary graphs, as this is essentially the same method used to count many other types of self-complementary structures. (The last section on sc-orbits provides a more general and direct method of enumerating self-complementary structures. We note in passing that a later attempt by D'Amore [102] to count self-complementary graphs was totally wrong; besides, it is not true that the necessary conditions he gave are also sufficient for a graph to be self-complementary). The problem was originally posed by Harary [175]. We normally partition graphs into isomorphism classes, and it was proved by Redfield [315], and also by Harary [173] using Pólya's enumeration theorem [289], that the generating function for isomorphism classes of graphs with n vertices (that is, the polynomial where the coefficient of x^k is the number of non-isomorphic graphs with n vertices and k edges) is

$$Z(S_n^{(2)}; 1+x, 1+x^2, 1+x^3, \ldots).$$

If we only want the total number of graphs, irrespective of the number of edges, then we just put $x = 1$ to get

$$g_n = Z(S_n^{(2)}; 2, 2, 2, \ldots).$$

We now define a new type of equivalence relation on graphs — two graphs G_1, G_2 are equivalent if $G_1 \cong G_2$ or $G_1 \cong \overline{G}_2$. This relation partitions graphs into *complementation classes*, whose number is given by de Bruijn [51] as a special case of his generalisation of Pólya's theorem:

$$c_n = \frac{1}{2} Z(S_n^{(2)}; 2, 2, 2, \ldots) + \frac{1}{2} Z(S_n^{(2)}; 0, 2, 0, 2, \ldots).$$

When $G \ncong \overline{G}$ the complementation class of G will correspond to two isomorphism classes, whereas if G is self-complementary its complementation class will correspond to just one isomorphism class. Thus if $\overline{g}_n$ denotes the number of sc-graphs, we have $g_n = 2c_n - \overline{g}_n$, so that

$$\overline{g}_n = 2c_n - g_n = Z(S_n^{(2)}; 0, 2, 0, 2, \ldots).$$

Similarly, for sc-digraphs we have

$$\overline{d}_n = Z(S_n^{[2]}; 0, 2, 0, 2, \ldots).$$

Wille [386], and later Xu, Wang and Wang [396], generalised the result for $\overline{g}_n$ by showing that the number of self-complementary r-multigraphs on n vertices is

$$\begin{aligned}\overline{g}_n^r &= Z(S_n^{(2)}; 0, r+1, 0, r+1, \ldots) \text{ when } r \text{ is odd, and}\\ \overline{g}_n^r &= Z(S_n^{(2)}; 1, r+1, 1, r+1, \ldots) \text{ when } r \text{ is even.}\end{aligned}$$

We can give more explicit formulas, both for the classic case ($r = 1$) and for the general case.

7.8. Theorem [Read 1963]. *The number of self-complementary graphs of even and odd orders is given, respectively, by*

$$\begin{aligned}\overline{g}_{4n} &= \sum_{\mathbf{j}} \frac{2^{c(\mathbf{j})}}{\prod k^{j_k} j_k!}\\ \overline{g}_{4n+1} &= \sum_{\mathbf{j}} \frac{2^{c(\mathbf{j})+\sum j_k}}{\prod k^{j_k} j_k!}\end{aligned}$$

where the sums are over all partitions $\mathbf{j}$ *of* n, *and*

$$c(\mathbf{j}) = 2\sum_{k=1}^{n} j_k(kj_k - 1) + 4 \sum_{1 \le r < t \le n} \langle r, t\rangle j_r j_t.$$

The number of self-complementary digraphs of even and odd orders is given, respectively, by

$$\begin{aligned}\overline{d}_{2n} &= \sum_{\mathbf{j}} \frac{2^{c(\mathbf{j})}}{\prod k^{j_k} j_k!}\\ \overline{d}_{2n+1} &= \sum_{\mathbf{j}} \frac{2^{c(\mathbf{j})+2\sum j_k}}{\prod k^{j_k} j_k!}.\end{aligned}$$

7.9. Theorem [Wille 1978]. *The number of self-complementary* r*-multigraphs on* n *vertices, for* r *odd, is given by*

$$\overline{g}_{4n+2}^r = \overline{g}_{4n+3}^r = 0$$

$$\overline{g}^r_{4n} = \sum \frac{(r+1)^{c'(\mathbf{j})}}{\prod k^{j_k} j_k!}$$

$$\overline{g}^r_{4n+1} = \sum \frac{(r+1)^{c'(\mathbf{j})+\sum_{i=1}^n j_{4i}}}{\prod k^{j_k} j_k!}$$

where the sums are over all partitions $\mathbf{j}$ *of* $4n$ *[resp.* $4n+1$*] with* $j_1 = 0$ *[resp.* $j_1 = 1$*] and all other* j_k *equal to* 0 *whenever* $k \not| 4$*; and where*

$$c'(\mathbf{j}) = \sum i = 1^n 2ij_{4i}^2 + 4 \sum_{1 \le s \le t \le n} [s,t] j_{4s} j_{4t}.$$

For r *even we have*

$$\overline{g}^r_n = \sum_{\mathbf{j}} \frac{(r+1)^{c''(\mathbf{j})}}{\prod k^{j_k} j_k!}$$

where the summation is over all partitions $\mathbf{j}$ *of* n*, and*

$$c''(\mathbf{j}) = \sum_i \sum_{\substack{s<t \\ \langle s,t\rangle = 2i}} [s,t] j_s j_t + \sum_{i=1}^{\lfloor n/2 \rfloor} (ij_{2i}^2 - j_{2i} + j_{4i})$$

7.10. It can be seen that $\overline{g}_{4n} = \overline{d}_{2n}$. Read noted this in his paper, and Morris discussed this interesting observation in [258], but to date no one has found a natural bijection between self-complementary graphs on $4n$ vertices and self-complementary digraphs on $2n$ vertices. Robinson [324, Section 6] and Clapham [88] obtained $\overline{g}_n$ and $\overline{d}_n$ by using Sachs' construction of self-complementary graphs. This method, which was suggested by Read in his review of Sachs' paper, is more direct but it does not seem to offer new insights into the identity above. The first few values of $\overline{g}_n$ and $\overline{d}_n$ are tabulated in 7.15 and 7.27, respectively.

7.11. Robinson [324] showed many other similar correspondences. He defined a *bilayered digraph* as a superposition of two digraphs, with the edges of one coloured red and the edges of the other coloured blue; the colours are not interchangeable, so that switching all red edges to blue, and *vice versa*, will not in general produce an isomorphic bilayered digraph. Then $\overline{d}_{2n} = b_n$, where b_n is the number of bilayered digraphs.

We can assign each vertex of a graph one of k colours; the number of graphs [self-complementary graphs] obtained this way is denoted by $g_n^{\{k\}}$ [$\overline{g}_n^{\{k\}}$]. If, moreover, we assign each vertex a $+$ or $-$ sign, we obtain a k-coloured signed graph; the total number of such graphs is denoted by $g_n^{[k]}$. The number of such graphs which are invariant under simultaneous complementation and sign reversal is denoted by $\overline{g}_n^{[k]}$ (our notation differs from that of Robinson). We can similarly use $d_n^{\{k\}}$, $t_n^{\{k\}}$, $r_n^{\{k\}}$,and $b_n^{\{k\}}$, for k-coloured digraphs, tournaments, relations and bilayered digraphs.

It is obvious that $g_n^{\{1\}}$, $\overline{g}_n^{\{1\}}$, $g_n^{[1]}$ and $\overline{g}_n^{[1]}$ are just the number of graphs, sc-graphs, marked graphs, and self-dual marked graphs (see 7.38 for definitions), respectively; with similar equivalences for other structures. Further, replacing each $+$ with a loop and each $-$ with no loop, we can see that $\overline{d}_{2n}^{[k]} = \overline{r}_{2n}^{\{k\}}$, while $\overline{g}_{4n}^{[k]}$ gives the number of k-coloured symmetric self-complementary relations on $4n$ elements and $\overline{t}_{2n}^{[k]}$ gives the number of k-coloured anti-symmetric self-complementary relations on $2n$ elements[1].

Robinson showed that $\overline{d}_{2n+1} = \overline{d}_{2n}^{[2]} = \overline{r}_{2n}^{\{2\}}$, $b_n^{[k]} = \overline{d}_{2n}^{[k]} = \overline{r}_{2n}^{\{k\}}$, $\overline{g}_{4n+1} = \overline{g}_{4n}^{[1]}$, and $\overline{t}_{2n+1} = \overline{t}_{2n}^{[1]}$.

For the natural bijections which explain the last two identities, see 1.40 and 5.9. Compare 7.23, 7.30 and 7.39, and see 7.59 for an alternating sum identity. Further results were found by Schwenk [183, eqns. 9, 14] (see [324, p. 174] for an interpretation) and Robinson [326].

7.12. Denoting the number of self-complementary blocks by $\overline{g}_n^b$, and the number of self-complementary graphs with exactly two end-vertices by $\overline{g}_n''$, Akiyama and Harary [12] proved the following (see Chapter 1 for more details):

Theorem. *For any positive integer $n \geq 4$ we have*

$$\begin{aligned}
\overline{g}_n &= \overline{g}_n'' + \overline{g}_n^b \\
\overline{g}_n'' &= \overline{g}_{n-4} \\
\overline{g}_n^b &= \overline{g}_n - \overline{g}_{n-4}. \square
\end{aligned}$$

[1] A *symmetric relation* is one where $a \to b$ iff $b \to a$; an *antisymmetric relation* is one where, for $a \neq b$, $a \to b$ iff $b \not\to a$. These concepts should not be confused with symmetric graphs, i.e. graphs which are both vertex- and edge-transitive.

7.13. Palmer [278] modified a method of Oberschelp [275] to find asymptotic formulas for $\overline{d}_n$ and $\overline{g}_n$. These methods were also used by Wille to obtain an asymptotic formula for self-complementary m-ary relations [385] (see 7.29) and self-complementary r-multigraphs, r odd [386].

Theorem [Palmer 1970, Wille 1978]**.** *The numbers of sc-digraphs, and sc-r-multigraphs with $r = 2k-1$ odd, are given asymptotically by*

$$\begin{aligned}
\overline{d}_{2n} = \overline{g}^{1}_{4n} &\sim \frac{2^{2n^2-2n}}{n!} \\
\overline{d}_{2n+1} &\sim \frac{2^{2n^2}}{n!} \\
\overline{g}^{2k-1}_{4n} &\sim \frac{(2k)^{2n^2}}{n!4^n} \\
\overline{g}^{2k-1}_{4n+1} &\sim \frac{(2k)^{2n^2+n}}{n!4^n}. \square
\end{aligned}$$

For r even, no such formula can be given (though Wille gives a complicated asymptotic analysis, which was further explored by Robinson [324, p.178–9]), but in the classic case of graphs and digraphs we can give more detailed expressions.

7.14. Theorem [Palmer 1970]**.** *The numbers of self-complementary graphs and digraphs satisfy*

$$\begin{aligned}
\overline{g}_{4n} = \overline{d}_{2n} &= \frac{2^{2n^2-2n}}{n!}\left(1 + n(n-1)2^{5-4n} + O\left(n^3/2^{6n}\right)\right) \\
\overline{g}_{4n+1} &= \frac{2^{2n^2-n}}{n!}\left(1 + n(n-1)2^{4-4n} + O\left(n^3/2^{6n}\right)\right) \\
\overline{d}_{2n+1} &= \frac{2^{2n^2}}{n!}\left(1 + n(n-1)2^{3-4n} + O\left(n^3/2^{6n}\right)\right). \square
\end{aligned}$$

7.15. Using 7.13 and 7.14 to calculate first and second approximations, respectively, Palmer calculated the first few values of $\overline{g}_n$ given in Table 7.1. Approximations for $\overline{d}_{2n+1}$ can also be obtained from those for $\overline{g}_{4n+1}$ since it can be seen that $\overline{d}_{2n+1}$ is asymptotic to $2^n\overline{g}_{4n+1}$.

n	1st approx.	2nd approx.	$\overline{g}_n$
4	1	1	1
5	2	2	2
8	8	10	10
9	32	36	36
12	682	715	720
13	5,461	5,589	5,600
16	699,050	703,147	703,760
17	11,184,811	11,217,579	11,220,000

Table 7.1: Self-complementary graphs

Palmer also noted that self-complementary graphs are "scarce" in relation to graphs with the same number of vertices; even among graphs with the appropriate number of vertices and edges, they still remain scarce. That is, if we denote the number of graphs with n vertices [resp. n vertices and k edges] by g_n [resp. $g_{n,k}$] then

$$\frac{\overline{g}_n}{g(n)} \to 0, \text{ and}$$

$$\frac{\overline{g}_n}{g(n, n(n-1)/4)} \to 0,$$

and similar results hold for digraphs.

7.16. We cannot use the number of edges as an enumeration parameter as all sc-graphs have $\frac{1}{2}\binom{n}{2}$ edges. However, Parthasarathy and Sridharan [285], counted self-complementary graphs and digraphs according to their degree sequence. This can be used to count the number of regular sc-graphs and -digraphs, thus answering a question posed by Colbourn and Colbourn [94]. However, there is as yet no counting formula for strongly regular sc-graphs; Rosenberg [331] showed how to generate systematically strongly regular sc-graphs by solving certain systems of $0-1$ equations. This approach was further developed by Mathon [247], but the enumeration of these graphs just for values of $n \leq 49$ (see 3.35) still took up a lengthy paper, albeit one with extensive results on antimorphisms, block valency matrices, and automorphism partitions. See also 7.32, and 4.11 for an application of the Parthasarathy-Sridharan formula.

Rao [308] counted sc-graphs on n vertices with circumference equal to n, $n-1$ and $n-2$, respectively. (There are no other possibilities, as he showed in [300]). If we denote the number of sc-graphs with n vertices and circumference $n-i$ by ${}_i\overline{g}_n$, then ${}_2\overline{g}_n$ is given by

$$ {}_2\overline{g}_{4k+\epsilon} = 1 + \sum_{i=1}^{k-1} {}_2\overline{g}_{4i+\epsilon}, \text{ for } \epsilon = 0, 1. $$

Moreover

$$ \lim_{k\to\infty} \frac{{}_2\overline{g}_{4k+\epsilon}}{\overline{g}_{4k+\epsilon}} = 0. $$

For the number of triangles in a sc-graph see 2.5, while for the number of degree sequences of sc-graphs see 6.4 and 6.17.

7.17. While labelled structures are generally easier to handle, the enumeration of labelled self-complementary graphs and digraphs has proved to be more difficult than the unlabelled case. Xu and Li [394] showed that the number of labelled sc-graphs with $4, 5, 8$ and 9 vertices are 12, 72, 112, 140 and 4, 627, 224, respectively; the values for $n = 4, 5$ are quickly checked. Ambrosimov [23] reports an asymptotic formula:

$$ \begin{aligned} \overline{G}_n &\sim (n/e)^{3n/4} 2^{n^2/8} \text{ for } n = 4k, \text{ and} \\ \overline{G}_n &\sim n^{1/4}(n/e)^{3n/4} 2^{(n^2-1)/8} \text{ for } n = 4k+1. \end{aligned} $$

However the values given by this approximation for $n = 4$, 5, 8 and 9 are 13, 118, 166, 346 and 5, 734, 688, respectively, which seems inconsistent with Xu and Li's figures.

Prime order vertex-transitive digraphs[2].

7.18. A vertex-transitive (di)graph is one where, for any two vertices u, v, there is an automorphism mapping u to v. A circulant (di)graph is one whose vertices can be numbered such that $(1, 2, \ldots, n)$ is an automorphism of the (di)graph. Turner [373] proved that a connected (di)graph on a prime number p of vertices is vertex-transitive if and only if it is circulant.

[2]See the note in 0.13

Alspach [20] showed that all circulant digraphs are self-converse, and thus vertex-transitive tournaments on p vertices are self-complementary. Astie [28] counted the vertex-transitive tournaments, while Chao and Wells [67] showed that the generating function for vertex-transitive digraphs[3] on p vertices, p prime, is

$$\frac{1}{p-1}\sum_{d|p-1}\phi(d)(1+x^{dp})^{(p-1)/d},$$

where $\phi(d)$ is the Euler function. For the first few primes we have the following generating functions:

$$\begin{aligned}
p &= 2, 1+x^2\\
p &= 3, 1+x^3+x^6\\
p &= 5, 1+x^5+2x^{10}+x^{15}+x^{20}\\
p &= 7, 1+x^7+3x^{14}+4x^{21}+3x^{28}+x^{35}+x^{42}\\
p &= 11, 1+x^{11}+5x^{22}+12x^{33}+22x^{44}+26x^{55}+\\
&\quad +22x^{66}+12x^{77}+5x^{88}+x^{99}+x^{110}
\end{aligned}$$

7.19. Chao and Wells also showed that the automorphism group of a non-trivial (i.e. non-null and non-complete) prime-order vertex-transitive digraph is $\Gamma_{\alpha,k} := \langle R, \sigma\rangle$, for some α, k, with the defining relations

$$R^p = 1,\ \sigma^\alpha = 1,\ \sigma^{-1}R\sigma = R^k,\ \text{where } \alpha|p-1 \text{ and } k^\alpha \equiv 1 \pmod p.$$

When $\alpha = 1$ we get a cyclic group, and the digraph is said to be *strongly vertex-transitive*, or svtsc-digraph if it is also self-complementary.

Chia and Lim [78] showed that vertex-transitive self-complementary digraphs (vtsc-digraphs) on p vertices are either tournaments or graphs; they also found many enumeration results for these graphs.

Let $\overline{c}^d_{p,\alpha}$ denote the number of (circulant) vtsc-digraphs with automorphism group $\Gamma_{\alpha,k}$ for some k. Then

$$\overline{c}^d_{p,\alpha} = \frac{2\alpha}{p-1}\sum_{\alpha||\beta|(p-1)/2}\mu\left(\frac{\beta}{\alpha}\right)2^{(p-1)/2\beta-1}$$

where $\alpha||\beta|(p-1)/2$ means that α divides β, β divides $(p-1)/2$, and $\beta/\alpha = 1$ or a product of distinct odd primes.

[3] In general, these are not self-complementary, even if they have $p(p-1)$ arcs.

7.20. Theorem[Chia and Lim 1986]. *The numbers of vtsc-graphs, vtsc-tournaments, vtsc-digraphs, svtsc-digraphs, and vtsc-digraphs with non-cyclic automorphism group, on a prime number p of vertices, denoted respectively by $\overline{c}_p^g$, $\overline{c}_p^t$, $\overline{c}_p^d$, $\overline{c}_p'$ and $\overline{c}_p''$ are given by:*

$$\begin{aligned}
\overline{c}_p^g &= \sum_{\substack{\alpha|(p-1)/2, \\ \alpha\, even}} \overline{c}_{p,\alpha}^d \\
\overline{c}_p^t &= \sum_{\substack{\alpha|(p-1)/2, \\ \alpha odd}} \overline{c}_{p,\alpha}^d \\
\overline{c}_p^d &= \sum_{\alpha|(p-1)/2} \overline{c}_{p,\alpha}^d \\
\overline{c}_p' &= \overline{c}_{p,1}^d \\
\overline{c}_p'' &= \sum_{\substack{\alpha|(p-1)/2, \\ \alpha\neq 1}} \overline{c}_{p,\alpha}^d
\end{aligned}$$

7.21. It follows from the definitions that $\overline{c}_p^g + \overline{c}_p^t = \overline{c}_p^d = \overline{c}_p' + \overline{c}_p''$. Their numbers are tabulated in Table 7.2 for $p \leq 41$.

p	$\overline{c}_p^g$	$\overline{c}_p^t$	$\overline{c}_p^d$	$\overline{c}_p'$	$\overline{c}_p''$
3	0	1	1	1	0
5	1	1	2	1	1
7	0	2	2	1	1
11	0	4	4	3	1
13	2	6	8	5	3
17	4	16	20	16	4
19	0	30	30	28	2
23	0	94	94	93	1
29	10	586	596	585	11
31	0	1096	1096	1091	5
37	30	7286	7316	7280	36
41	56	26216	26272	26214	58

Table 7.2: Self-complementary circulant graphs and digraphs

7.22. There are a number of different expressions for these quantities, such as the following by Chia and Lim:

$$\overline{c}'_p = \frac{1}{p-1} \sum_{\substack{\alpha|(p-1)/2, \\ \alpha \text{ odd}}} \mu(\alpha) 2^{(p-1)/2\alpha}$$

where $\mu(n)$ denotes the classical Möbius function,

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ (-1)^k & \text{if } n \text{ is the product of } k \text{ distinct primes} \\ 0 & \text{otherwise} \end{cases}$$

Astie [28] (also quoted in [78, Thm. 4(i)]) and Chao and Wells [68] both gave further, more complicated, expressions for the number of strongly vertex-transitive digraphs, while the latter authors gave the following formulation for the number of vtsc-digraphs:

$$\overline{c}^d_p = \frac{1}{p-1} \left(\sum_{d|p-1} \phi(d) \frac{\partial^{(p-1)/d}}{\partial z_d^{(p-1)/d}} \right) e^{2(z_2+z_4+\cdots)},$$

evaluated at $z_1 = z_2 = \cdots = 0$, where ϕ is the Euler function. In his review, Alspach combined their methods with those of his own paper [21] to give the following result: the number of vtsc-digraphs on p vertices, whose automorphism group is the transitive group of degree p and order hp, $h < p-1$ is

$$\sum_{d|(p-1)/h} \mu(d) Z \left(C_{(p-1)/hd}; \frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \cdots \right) e^{2(z_2+z_4+\cdots)}$$

evaluated at $z_1 = z_2 = \cdots = 0$.

We also note that Alspach gave a different expression for $\overline{c}^t_p$ in another paper [20].

7.23. Later on, Klin, Liskovets and Pöschel [222] expressed $\overline{c}^g_p$, $\overline{c}^t_p$ and $\overline{c}^d_p$ in terms of cycle indices, for odd primes:

$$\begin{aligned} \overline{c}^g_p &= Z(C_{(p-1)/2}; 0, 2, 0, 2, \ldots) \\ \overline{c}^t_p &= Z(C_{p-1}; \sqrt{2}, 0, \sqrt{2}, 0, \ldots) \\ \overline{c}^d_p &= Z(C_{p-1}; 0, 2, 0, 2, \ldots).\square \end{aligned}$$

This gives us an interesting counterpart to 7.10.

Corollary[Klin, Liskovets and Pöschel 1996]**.** *If p is an odd prime number such that $2p-1$ is also prime, then*

$$\overline{c}^{\,g}_{2p-1} = \overline{c}^{\,d}_{p}.\Box$$

7.24. Chao and Wells [68, Thm. 2] counted the circulant sc-digraphs on pq vertices, where p and q are distinct primes, while Klin, Liskovets and Pöschel did the same for sc-digraphs on p^2 vertices.

Theorem. *For any odd prime p,*

$$\begin{aligned}
\overline{c}^{\,g}_{p^2} &= \mathcal{C}^*(p^2; 0,2,0,2,\ldots; 0,2,0,2,\ldots),\\
\overline{c}^{\,t}_{p^2} &= \mathcal{C}(p^2; \sqrt{2},0,\sqrt{2},0,\ldots; \sqrt{2},0,\sqrt{2},0,\ldots),\\
\overline{c}^{\,d}_{p^2} &= \mathcal{C}(p^2; 0,2,0,2,\ldots; 0,2,0,2,\ldots),
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{C}(p^2;\mathbf{x};\mathbf{y}) &:= \frac{1}{p}Z(Z_{p-1};\mathbf{x}^{p+1}) - \frac{1}{p}Z(Z_{p-1};\mathbf{xy}) + Z(Z_{p-1};\mathbf{x})Z(Z_{p-1};\mathbf{y}),\\
\mathcal{C}^*(p^2;\mathbf{x};\mathbf{y}) &:= \frac{1}{p}Z(Z_{\frac{p-1}{2}};\mathbf{x}^{p+1}) - \frac{1}{p}Z(Z_{\frac{p-1}{2}};\mathbf{xy}) + Z(Z_{\frac{p-1}{2}};\mathbf{x})Z(Z_{\frac{p-1}{2}};\mathbf{y}),
\end{aligned}$$

and $\mathbf{xy} := (x_1y_1, x_2y_2, \ldots)$. *More explicitly,*

$$\begin{aligned}
\overline{c}^{g}_{p^2} &= \frac{2}{p(p-1)} \sum_{\substack{2r|(p-1),\\ r\ \mathit{even}}} \phi(r)\left(2^{(p+1)^{(p-1)/2r}} - 2^{2^{(p-1)/2r}}\right) + \\
&\quad + \left[\frac{2}{p-1} \sum_{\substack{2r|(p-1),\\ r\ \mathit{even}}} \phi(r) 2^{(p-1)/2r}\right]^2 \\
\overline{c}^{t}_{p^2} &= \frac{1}{p(p-1)} \sum_{\substack{r|(p-1),\\ r\ \mathit{odd}}} \phi(r)\left(2^{(p+1)^{(p-1)/2r}} - 2^{2^{(p-1)/2r}}\right) + \\
&\quad + \left[\frac{1}{p-1} \sum_{\substack{r|(p-1),\\ r\ \mathit{odd}}} \phi(r) 2^{(p-1)/2r}\right]^2 \\
\overline{c}^{d}_{p^2} &= \frac{1}{p(p-1)} \sum_{\substack{r|(p-1),\\ r\ \mathit{even}}} \phi(r)\left(2^{(p+1)^{(p-1)/r}} - 2^{2^{(p-1)/r}}\right) + \\
&\quad + \left[\frac{1}{p-1} \sum_{\substack{r|(p-1),\\ r\ \mathit{even}}} \phi(r) 2^{(p-1)/r}\right]^2 . \square
\end{aligned}$$

7.25. Finally Klin, Liskovets and Pöschel [222]tabulated some values and proved a number of identities, among which are the following:

Proposition. *Let* c^t_n *denote the number of circulant tournaments on* n *vertices. Then for* p *an odd prime,* $p \equiv 3 \pmod 4$,

$$\begin{aligned}
\overline{c}^{g}_{p^2} &= 0 \\
\overline{c}^{d}_{p^2} &= c^t_{p^2} . \square
\end{aligned}$$

The first identity is a special case of a result by Sachs [341] who proved that there are no self-complementary circulant graphs on p^{2r} vertices when p is a prime congruent to $3 \pmod 4$, and of Fronček, Rosa and Širáň [129] who showed that self-complementary circulant graphs of order n exist iff every prime divisor of n is congruent to 1 (mod 4). Since every circulant tournament is self-complementary, the proposition states that every circulant self-complementary digraph on p^2 vertices is a tournament. Some values of $\overline{c}^g_{p^2}$ and $\overline{c}^d_{p^2}$ are tabulated in Table 7.3.

p	$\overline{c}^g_{p^2}$	$\overline{c}^d_{p^2}$
3	0	3
5	7	214
7	0	399,472
11	0	-
13	56,385,212,104	-
17	-	-
19	0	-

Table 7.3: Prime square circulant sc-graphs and -digraphs

Relations and self-converse digraphs

7.26. Unlike self-complementary digraphs, self-converse digraphs can have any number of edges, so now for each n we want a counting polynomial ${d'}_n(x)$ which has as the coefficient of x^k the number of self-converse digraphs on n vertices and k edges. For example it can be checked that the counting polynomial for self-converse digraphs on 3 vertices is

$$ {d'}_3(x) = 1 + x + 2x^2 + 2x^3 + 2x^4 + x^5 + x^6. $$

(See Figure 5.1). A symmetry can be seen in the coefficients, and this will be true for all ${d'}_n(x)$ because $(\overline{D})' = \overline{D'}$.

We also want a counting polynomial for self-converse digraphs with loops permitted, that is, self-converse relations. The required results were given by Harary and Palmer [180] (see also 7.38):

Theorem. *The counting polynomial for self-converse digraphs on n vertices is given by*

$$ {d'}_n(x) = 2Z(S_n^{S_2 *}, 1 + x) - Z(S_n^{[2]}, 1 + x). $$

The counting polynomial for self-converse relations on n elements is given by

$$r'_n(x) = 2Z(S_n^{S_2}, 1+x) - Z(S_n^2, 1+x).\Box$$

Harary and Palmer gave more explicit but much more complex formulas for $d'_n(x)$ and $r'_n(x)$ in [180] and also in [182, pp.152–4].

7.27. Harary and Palmer also gave a formula for the total number of self-converse digraphs on n vertices, and tabulated this, and the number of self-converse relations and self-complementary digraphs (Table 7.4). The values for $n = 6$ are given differently in their book [182, p. 155, 243]; we reproduce this version here as it is more recent.

Theorem. *The total number of self-converse digraphs on n vertices is given by*

$$d'_n = d'_n(1) = \frac{1}{n!} \sum_{\alpha \in S_n} 2^{\varepsilon(\alpha)}$$

where

$$\begin{aligned} \varepsilon(\alpha) &= \sum_{k=1}^{n} \left[\langle 2, k \rangle \left\{ \frac{k-1}{2} j_k + k \binom{j_k}{2} \right\} + \eta(k) j_k \right] \\ &\quad + \sum_{1 \le r < t \le n} \langle 2, [r,t] \rangle \langle r, t \rangle j_r j_s . \Box \end{aligned}$$

7.28. Robinson [325] gave an asymptotic expression for the number of self-converse digraphs (this corrects a mistaken result given by Sridharan [358]):

$$d'_n \sim \frac{2^{(n^2-n)/2}}{n!} \left(\frac{2n}{e} \right)^{n/2} \frac{e^{\sqrt{n/2}}}{e^{1/8}\sqrt{2}}$$

7.29. Wille [385] tackled m-ary relations over a finite set $N = \{1, \ldots, n\}$. We can think of these as a set of ordered m-tuples, or as directed hypergraphs, with repeated vertices allowed in each edge. The complement of an m-ary relation R is $\overline{R} = N^m - R$, and self-complementary m-ary relations are defined in the obvious way. We now need a new permutation group, S_n^m,

n	$\overline{d}_n$	d'_n	r'_n
1	1	1	2
2	1	3	8
3	4	10	44
4	10	70	436
5	136	708	7,176
6	720	15,248	222,368
7	44,224	543,520	
8	703,760		

Table 7.4: Self-complementary digraphs, and self-converse digraphs and relations

which is the group of permutations induced on m-tuples by S_n. We note that since an m-tuple can contain repeated elements $S_n^2 \neq S_n^{[2]}$.

Theorem. *The number $\overline{r}_n^m$ of self-complementary m-ary relations on n elements is given by*

$$\begin{aligned}\overline{r}_{2n+1}^m &= 0\\ \overline{r}_{2n}^m &= Z(S_{2n}^m; 0, 2, 0, 2, \ldots) = \sum_{\mathbf{j}} \frac{2^{\sum s_{2k}}}{\prod k^{j_k} j_k!}\end{aligned}$$

where the summation is over all partitions $\mathbf{j}$ of $2n$ with $j_k > 0$ only when k is even, and

$$s_{2k} = \frac{1}{2k} \sum_{\langle r,s\rangle = 2k} [r,s] j_r j_s.$$

Asymptotically,

$$\overline{r}_{2n}^m \sim \frac{2^{(2n)^m/2}}{n!2^n}.\Box$$

7.30. Corollary. $\overline{r}_{2n}^2 = \overline{g}_{4n+1}$. $\Box$

7.31. A mixed graph is a graph which can contain both ordinary and oriented edges. If we consider an ordinary edge to be a symmetric pair of directed edges, we see that the mixed graphs on n vertices are just the digraphs on n vertices, but we can now count them with the number of symmetric pairs as an enumeration parameter. Sridharan [356] gave the counting formula for mixed self-complementary graphs, $\overline{m}_1(x,y)$, and mixed self-converse graphs, $\overline{m}_2(x,y)$, where the coefficient of $x^r y^t$ is the number of mixed graphs with r directed edges and t ordinary edges.

$$\begin{aligned}
\overline{m}_1(x,y) &= \frac{1}{n!}\sum_{\gamma\in S_n}\prod_{k\,\text{even}} 2^{kj_k{}^2/2}(x^k+y^{k/2})^{j_k(kj_k-2)/2}x^{kj_k\eta(k)/2}y^{kj_k(1-\eta(k))/2}\\
&\quad \prod_{1\le r<t\le n}\left[2(x^{[r,t]}+y^{[r,t]/2})\right]^{\langle r,t\rangle j_r j_t}\\
\overline{m}_2(x,y) &= \frac{1}{n!}\sum_{\gamma\in S_n}\prod_{k\,\text{odd}} b_{2k}^{(kj_k-1)j_k/2}\prod_{k\,\text{even}} a_k^{(kj_k-2)j_k/2}a_{k/2}^{\eta(k)j_k}b_k^{(1-\eta(k))j_k}\\
&\quad \prod_{\substack{1\le r<t\le n\\ r\text{ and }t\\ \text{both odd}}} b_{2[r,t]}^{\langle r,t\rangle j_r j_t}\prod_{\substack{1\le r<t\le n\\ r\text{ and }t\text{ not}\\ \text{both odd}}} a_{[r,t]}^{\langle r,t\rangle j_r j_t}
\end{aligned}$$

where

$$\begin{aligned}
a_k &= 1+2x^k+y^k,\\
b_k &= 1+y^{k/2},
\end{aligned}$$

and

$$\eta(k)=\begin{cases}1 & \text{if } \frac{k}{2} \text{ is odd}\\ 0 & \text{otherwise}\end{cases}$$

Special types of graphs and digraphs

7.32. An oriented graph is a digraph where no two vertices x and y may be joined by both (x,y) and (y,x); alternatively, we may look at it as a mixed graph with no ordinary edges. Harary and Palmer asked for the number of self-converse oriented graphs in [180]. Sridharan [355] solved this problem, and also counted the self-complementary oriented graphs, which are just the sc-tournaments.

Theorem. *The number of sc-tournaments is*

$$\bar{t}_n=\frac{1}{n!}\sum_{\alpha\in B_n}2^{O_n(\alpha)}$$

where

$$\begin{aligned}
B_n &= \{\alpha \in S_n : \alpha \textit{ has type}(1^{j_1}2^{j_2}6^{j_6}10^{j_{10}}\cdots),\ \textit{with } j_1 = 1 \textit{ or } 0\}, \\
O_n(\alpha) &= \sum_{p\in N} \frac{p}{2} {j_p}^2 + \sum_{q<r\in N'} \langle q,r\rangle j_q j_r, \\
N &= \{2,6,10,14,\ldots\} \textit{ and } N' = \{1,2,6,10,14,\ldots\}.\Box
\end{aligned}$$

Hegde and Sridharan [203] counted self-complementary oriented graphs and self-converse digraphs according to their degree sequence. This gives us a way of counting regular sc-tournaments, thus solving a problem posed by Colbourn and Colbourn [94]; see also 7.16.

7.33. Harary and Palmer [182, p. 156] gave a more explicit formula for sc-tournaments on $2n$ vertices. Eplett [112] gave a formula for the number of sc-tournaments on $2n+1$ vertices, and noted that both formulas could be written in terms of partitions of n (rather than partitions of $2n$ or $2n+1$). Thus,

$$\begin{aligned}
\bar{t}_{2n} &= \sum \frac{\prod_k 2^{kj_k^2 - j_k} \prod_{r<t} 2^{2\langle r,t\rangle j_r j_t}}{\prod_k k^{j_k} j_k!} \\
\bar{t}_{2n+1} &= \sum \frac{\prod_{r,t=1}^{n} 2^{\langle r,t\rangle j_r j_t}}{\prod_k k^{j_k} j_k!}
\end{aligned}$$

where the summations are over all partitions $\mathbf{j}$ of n with $j_k > 0$ only when k is odd.

7.34. Harary and Palmer [182, p. 215], and later Sridharan [357] gave an asymptotic formula for the number of sc-tournaments.

Theorem. *The number of sc-tournaments on $2n$ or $2n+1$ vertices satisfy*

$$\bar{t}_{2n} = \frac{2^{n^2-n}}{n!}\left[1+\frac{n(n-1)(n-2)}{3}2^{8-4n}+O\left(\frac{n^5}{2^{20n/3}}\right)\right]$$

$$\bar{t}_{2n+1} = \frac{2^{n^2}}{n!}\left[1+\frac{n(n-1)(n-2)}{3}2^{6-4n}+O\left(\frac{n^5}{2^{20n/3}}\right)\right].\square$$

Using $2^{n^2-n}/n!$ and $2^{n^2}/n!$ for a first approximation to $\bar{t}_{2n}$ and $\bar{t}_{2n+1}$ respectively, we have the values given in Table 7.5 for the number of sc-tournaments.

n	1st approx.	2nd approx.	$\bar{t}_n$
3	2	2	2
4	2	2	2
5	8	8	8
6	11	12	12
7	85	88	88
8	171	176	176
9	2,731	2,758	2,752
10	8,738	8,826	8,784
11	279,620	279,962	279,968

Table 7.5: Sc-tournaments

Comparing with the results of 7.13 we see that sc-tournaments are scarce even among self-complementary digraphs.

7.35. Sridharan [358] also attempted to give an asymptotic formula for the number of self-converse oriented graphs but Palmer pointed out an error in his review, and showed that the formula for $n = 20$ is already too low by a factor of at least 10^{10}. The problem is that, instead of a single dominant term, there are an infinite family of terms to consider in the asymptotic treatment. Robinson [322] did in fact manage to give an asymptotic estimate, but it is much more complicated than those for other types of graphs. He also tabulated exact values of the number of self-converse oriented graphs up to $n = 27$.

7.36. A digraph is said to be transitive if the existence of the arcs (a, b) and (b, c) always implies the existence of the arc (a, c). Hegde, Read and Sridharan [202] found the structure of transitive self-complementary digraphs (see 5.10) and counted them, even giving an explicit formula:

Theorem. *The number a_n of transitive self-complementary digraphs on $2n$ vertices is the same as the number on $2n+1$ vertices. It is given either by the generating function*

$$\frac{1-x}{1-2x-x^2},$$

or by the recurrence relation

$$a_n = 2a_{n-1} + a_{n-2}$$

with initial conditions

$$a_0 = a_1 = 1,$$

or by the formula

$$a_n = \frac{1}{2}\left\{\left(1+\sqrt{2}\right)^n + \left(1-\sqrt{2}\right)^n\right\}. \square$$

7.37. Quinn [291] gave the generating function for $\overline{g}_{n_1,n_2}$, the number of non-isomorphic self-complementary bipartite graphs with parts of order n_1 and n_2, and listed these numbers for $n_1 \leq 3$ and $n_2 \leq 4$ (Table 7.6). An asymptotic expression is given in [294].

n_1 \ n_2	1	2	3	4
1	0	1	0	1
2	1	2	3	6
3	0	3	0	7

Table 7.6: Bipartite self-complementary graphs with parts of size n_1, n_2

7.38. We now give some definitions that will be useful later on. A signed [marked] graph is one which has a + or − sign on each of its edges [vertices].

A net is a graph where both edges and vertices are signed. The dual of any of these types of graphs is obtained by changing all the signs. A signed graph, marked graph or net is self-dual if it is isomorphic to its dual. For nets, we also define the edge-dual [vertex-dual], obtained by changing only the signs on the edges [vertices]; if this is isomorphic to the original net, we say that it is edge-self-dual [vertex-self-dual]. Thus a net can have three types of self-duality; a net possessing any two must also possess the third, and so we say that it is doubly self-dual.

Harary, Palmer, Robinson and Schwenk [183] counted all these types of structures, except for the doubly self-dual nets; they also provided asymptotics and tabulated exact values up to $n = 12$. Bender and Canfield [35] re-derived all their results as applications of a more general theorem, and showed that in each case almost all the graphs are connected, and almost all the disconnected graphs have just two components, one of them an isolated vertex. Their results also generalize the enumeration of self-converse digraphs, isographs, mixed graphs and oriented graphs, and the colour cyclic factorisations of Schwenk described below. Read essentially counted self-dual signed graphs in [314].

Holroyd [208] managed to enumerate doubly self-dual acyclic and unicyclic nets, but the general case remains open. Palmer and Schwenk [283] counted necklaces (signed circuits) which are rotationally equivalent to their dual and their reflection (i.e. there is a rotational automorphism taking them to their dual, and a rotational automorphism taking them to their reflection). In general, however, enumerating structures with two different self-dualities is much more difficult than for structures with just one self-duality. The class of digraphs which are both self-converse and self-complementary is one notable open case. Palmer [280] did manage to count the digraphs whose converse and complement are isomorphic:

$$\overline{d}'_n = 2Z(S_n^{S_2*}; 0, 2, 0, 2, \ldots) - Z(S_n^{[2]}; 0, 2, 0, 2, \ldots).$$

However, this is a problem with two dualities and not self-dualities.

7.39. The sign of any subgraph of a signed graph is defined to be the product of the signs of all its edges. If every circuit has positive sign, the graph is said to be balanced. Motivated by this definition, Sozański [354] considered the concept of weak isomorphism — two signed graphs G_1, G_2 are weakly isomorphic if there is an isomorphism between the underlying graphs of G_1

and G_2 which preserves the sign of all the circuits. A signed graph is then said to be weakly self-dual if it is weakly isomorphic to its dual.

Theorem. *The number of weak isomorphism classes of complete signed graphs, and weak isomorphism classes of weakly self-dual complete signed graphs on n vertices, respectively, is given by*

$$\begin{aligned} w_n &= \frac{1}{n!}\sum_{\alpha\in S_n} 2^{I_2(\alpha)-I_1(\alpha)} \\ \overline{w}_n &= \frac{1}{n!}\sum_{\alpha\in M_n} 2^{I_2(\alpha)-I_1(\alpha)} \end{aligned}$$

where α is a permutation with k cycles, $I_1(\alpha) = k-1$ or k depending on whether α does or does not have an odd length cycle, $I_2(\alpha)$ is the number of cycles of the permutation which α induces on edges, and M_n is the set of permuations in S_n that either have all cycles of even length, or have 1 or 2 cycles of length 1 and all other cycles of length divisible by 4. For $n \equiv 3 \pmod 4$, $\overline{w}_n = 0$. □

7.40. Now w_n is the number of Eulerian graphs [321], which is also the number of switching classes [243] and the number of two-graphs [351] on n vertices. A two-graph is a set of triples from $\{1, 2, \ldots, n\}$ such that any four vertices contain an even number of triples. A switching class is an equivalence class under the operation of switching at a vertex; see 4.28 for a definition of switching, and its use as a self-complement index. A natural bijection with Eulerian graphs is known only for even n, as in this case each switching class contains exactly one Eulerian graph.

Meanwhile, $\overline{w}_n$ is equal to the number of sc-two-graphs and, for $n = 4k+1$, it is equal to the number of Eulerian sc-graphs on n vertices, which Robinson [321] showed to be in a natural one-one correspondence with the sc-graphs on $4k$ vertices (see 1.40 for the proof). Maybe these correspondences will help to throw some light on the problems of 7.10, 7.23 and 7.30.

7.41. The situation for Eulerian digraphs is more complex than for undirected graphs. An isograph is a digraph in which, at each vertex, the indegree is equal to the outdegree. An Eulerian digraph is then just a weakly connected isograph; in particular, Eulerian self-complementary digraphs are just the self-complementary isographs. In [361] Sridharan and Parthasarathy show how to count, for each n,

(a) self-converse isographs with a given number of edges

(b) Eulerian self-converse isographs with a given number of edges

(c) self-complementary isographs

and the oriented versions of these three types of graphs. The procedure is quite complex and they do not provide explicit formulas. In his review, Robinson pointed out that the methods proposed for (a) actually count just those isographs each of whose components is self-converse; and that Harary and Prins' methods [184] could be used to get the total number of self-converse isographs.

Colour cyclic factorisations

7.42. In a remarkable paper, Schwenk [348] managed to derive formulas which contain many of the results listed above as special cases. (His results are all stated for simple graphs, but analogous results hold for directed graphs too, and they can probably be extended to many other structures without difficulty). We start by giving his definition of colour cyclic factorisations. For an arbitrary graph G, a partition of its edges into k colour classes labelled $0, 1, \ldots, k-1$ is called a colour cyclic factorisation if G has an automorphism which cycles the colours, that is, each edge of colour i is mapped to an edge of colour $i+1 \pmod{k}$. When $G = K_n$ we get cyclically k-complementary graphs studied in 5.51–5.59; in particular, when $G = K_n$ and $k = 2$ we have the usual definition of self-complementary graphs. Note that

- the term "cyclic decomposition" has been used by many authors to mean something quite different from colour cyclic factorisation

- not all isomorphic factorisations are cyclic, as the group which permutes the k isomorphic subgraphs does not always contain C_k. Schwenk mentioned in particular the case where this group is the Klein four-group, $\{I,\ (AB),\ (CD),\ (AB)(CD)\}$, because it appears in enumeration problems involving two self-dualities, as discussed in 7.38. In [17] an infinite class of non-cyclic factorisations of K_{9k+10} into 9 isomorphic subgraphs is constructed.

7.43. Schwenk introduced the vector $\overline{m}_k$ to denote a vector whose ith entry is 1 whenever i is a multiple of k, and 0 otherwise. The vector $\overline{m}_k$ selects those variables whose subscripts are multiples of k. It can be multiplied by a scalar, for example $2\overline{m}_3 = (0, 0, 2, 0, 0, 2, \dots)$. As defined, $\overline{m}_k$ is an infinite vector, but it is understood to be appropriately truncated when used in a cycle index. That is, in $Z(\Gamma; \overline{m}_k)$, where $|\Gamma| = n$, we truncate $\overline{m}_k$ at the nth position.

The normal automorphism group which permutes the vertices of a graph G can be denoted by $Aut_V(G)$. In a natural way, $Aut_V(G)$ induces a group of permutations on the edges of G, which we denote by $Aut_E(G)$. In fact Sabidussi [339] and Harary and Palmer [181] proved that $Aut_V(G) \cong Aut_E(G)$ if and only if G has at most one isolated point, and does not contain K_2 as a component. (By abusing notation we can state this as follows: $Aut_V(G) \cong Aut_E(G)$ if and only if G does not contain K_2 or $\overline{K}_2$ as a component).

We can now state Schwenk's result.

7.44. Theorem. *The number of colour cyclic factorisations of a graph G using k colours is $Z(Aut_E(G); k\overline{m}_k)$.* □

7.45. Corollary [Harary, Palmer, Robinson and Schwenk 1977]**.** *The number of self-dual signed graphs whose underlying graph is G is $Z(Aut_E(G); 2\overline{m}_2)$.*

7.46. Corollary [Read 1963]**.** *The number of self-complementary graphs with n vertices is $Z(S_n^{(2)}; 2\overline{m}_2$.* □

7.47. A colour cyclic pattern is defined analogously to colour cyclic factorisations, except that we now colour the vertices instead of the edges.

Theorem [Schwenk 1984]**.** *The number of colour cyclic patterns for colouring the vertices of G using k colours is $Z(Aut_V(G);\ k\overline{m}_k)$.* □

7.48. Corollary [Harary, Palmer, Robinson and Schwenk 1977]**.**
The number of self-dual marked graphs whose underlying graph is G is

$Z(Aut_V(G); 2\overline{m}_2)$. $\square$

7.49. Schwenk then considered the problem where both vertices and edges are coloured. Let the vertices of G be coloured with j labels, and the edges with k labels. This colouring is called colour cyclic if there is an automorphism of G which cycles the vertex and edge labellings in tandem, that is, it adds 1 to each vertex label and to each edge label. We are not allowed to cycle vertex or edge labels independently, while holding the other set fixed.

A new cycle index is needed for this problem. Denoted

$$Z(Aut_{V,E}(G); x_1, x_2, \ldots, x_n; y_1, y_2, \ldots, y_m)$$

this cycle index has a term for each automorphism γ formed by multiplying the corresponding terms from $Z(Aut_V(G); x_1, x_2, \ldots, x_n)$ and from $Z(Aut_E(G); y_1, y_2, \ldots, y_m)$.

Theorem [Schwenk 1984]**.** *The number of colour cyclic patterns of G using j colours for the vertices and k colours for the edges is $Z(Aut_{V,E}(G); j\overline{m}_j; k\overline{m}_k)$.*

7.50. Corollary [Harary, Palmer, Robinson and Schwenk 1977]**.** *The number of self-dual nets with underlying graph G is $Z(Aut_{V,E}(G); 2\overline{m}_2; 2\overline{m}_2)$.* $\square$

7.51. Schwenk then generalised further by considering what he termed K-invariant colourings. Let K be a group of the form $Z_{k_1} \times Z_{k_2} \times \cdots \times Z_{k_m}$, with $\langle k_i, k_j \rangle = 1$ for all $i \neq j$. If the edges of G are coloured with $k_1 + k_2 + \cdots + k_m$ colours, partitioned into orbits of size $k_1, k_2, \ldots, k_m$, such that the colours in each orbit are cyclically equivalent, then we call this a K-invariant colouring. K is called a colour group.

The condition $\langle k_i, k_j \rangle = 1$ implies that every permutation of the colours can be achieved by repeatedly applying some automorphism of G to the original colouring. It is not known how to tackle the case where $\langle k_i, k_j \rangle \neq 1$ for some i, j.

Theorem [Schwenk 1984]**.** *If $\langle k_i, k_j \rangle = 1$ for all $i \neq j$, then the number of $Z_{k_1} \times Z_{k_2} \times \cdots \times Z_{k_m}$ colour invariant factorisations of G is*

$$Z(Aut_E(G); \sum_{i=1}^{n} k_i \overline{m}_{k_i}). \square$$

7.52. The colour group applied to the complete graph K_n can be viewed as a factorisation using k cyclically equivalent colours and one distinctive colour. When the distinctive colour is thought of as an invisible or absent edge, then each such colouring of K_n is just a k-colour cyclic factorisation of some graph on n vertices.

Corollary. *The total number of k-colour cyclic factorisations of all graphs on n vertices is given by $Z(S_n^{(2)}; \overline{m}_1 + k\overline{m}_k)$. In particular [Harary, Palmer, Robinson and Schwenk 1977], the number of self-dual signed graphs on n vertices is given by $Z(S_n^{(2)}; \overline{m}_1 + 2\overline{m}_2)$.* □

7.53. Schwenk then extended Palmer's methods [278] to give asymptotic formulas for the number of k-colour cyclic factorisations of K_n. Putting $k = 2$ in this result will give Palmer's asymptotic formula for the number of self-complementary graphs. Note that the four cases below cover all the values of n for which k-colour cyclic factorisations exist.

Theorem. *The asymptotic number of k-colour cyclic factorisations of the complete graph K_n is given by*

A. If k is even and $n = 2mk$,

$$Z(S_n^{(2)}; k\overline{m}_k) = \frac{k^{m^2k-m}}{m!2^m}\left(1 + m(m-1)2^{2k+1-2mk} + O\left(m^3/k^{3mk}\right)\right).$$

B. If k is even and $n = 2mk + 1$,

$$Z(S_n^{(2)}; k\overline{m}_k) = \frac{k^{m^2k}}{m!2^m}\left(1 + m(m-1)k^{2k-2mk} + O\left(m^3/k^{3mk}\right)\right).$$

C. If k is odd and $n = mk$,

$$Z(S_n^{(2)}; k\overline{m}_k) = \frac{k^{(m^2k-3m)/2}}{m!}\left(1 + \binom{m}{2}k^{k+2-mk} + O\left(m^3/k^{2mk}\right)\right)$$

D. If k is odd and $n = mk + 1$,

$$Z(S_n^{(2)}; k\overline{m}_k) = \frac{k^{(m^2k-m)/2}}{m!}\left(1 + \binom{m}{2}k^{k+1-mk} + O\left(m^3/k^{2mk}\right)\right). \square$$

k		sc-graphs								
n	1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1	1
2	1									
3	1		1							
4	1	1	3							
5	1	2			5					
6	1		18		25					
7	1		135				49			
8	1	10		32			343			
9	1	36	3,411	128					729	
10	1		89,694		39,375				6,561	

Table 7.7: k-colour cyclic factorisations of K_n

The exact values of $Z(S_n^{(2)}; k\overline{m}_k)$ are given in Table 7.7for $n, k \leq 10$.

7.54. It is also possible to obtain asymptotic formulas for the number of k-colour cyclic factorisations among all graphs with n vertices, that is, to estimate $Z(S_n^{(2)}; \overline{m}_1 + k\overline{m}_k)$. Robinson did this for the case $k = 2$ (self-dual signed graphs) but his formula takes up a full page of [183]. This complexity discouraged Schwenk from treating the general case. As above, however, he tabulated the exact numbers for $n \leq 10$ (Table 7.8).

Self-complementary k-plexes

7.55. A simplicial complex of order n consists of a finite non-empty set V of n points and a collection of subsets of V called simplexes such that every point is a simplex and every non-null subset of a simplex is a simplex. The dimension of a simplex S is $|S| - 1$; the dimension of a complex is the maximum dimension of its simplexes. The 0-dimensional complexes are just null graphs, while 1-dimensional complexes are just the non-null graphs. A simplicial complex where every maximal simplex has dimension k is said to be pure or homogeneous.

In [279] Palmer considered k-plexes, where every maximal simplex has

n \ k	graphs 1	self-dual signed graphs 2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1	1
2	2	1	1	1	1	1	1	1	1	1
3	4	2	2	1	1	1	1	1	1	1
4	11	6	6	2	1	1	1	1	1	1
5	34	20	14	7	8	2	1	1	1	1
6	156	86	93	32	44	10	1	1	1	1
7	1,044	662	993	107	152	61	74	1	1	2
8	12,346	8,120	7,965	903	404	405	586	92	1	2
9	274,668	171,526	151,152	13,498	908	2,809	2,634	821	1,112	5
10	12,005,168	5,909,259	6,859,540	309,378	204,138	14,830	8,778	7,382	11,112	1,469

Table 7.8: k-colour cyclic factorisations of all graphs

dimension k or 0 ($0 < k < n$). The complement of a k-plex K of order n is also a k-plex of order n denoted by $\overline{K}$. It has the same set of points as K, and its set of k-simplexes consists precisely of those which are absent from K. Thus the self-complementary 1-plexes are just the (non-trivial) self-complementary graphs. It is easily seen that all self-complementary k-plexes are connected (compare 1.8), and thus pure; and that there is a natural bijection between (self-complementary) $k+1$-uniform hypergraphs of order n and (self-complementary) k-plexes.

The appropriate permutation group for k-plexes of order n is $S_n^{(k+1)}$, the group of permutations induced by S_n on the $k+1$ subsets of $\{1, 2, \ldots, n\}$. Palmer's result is then as follows:

Theorem. *The counting polynomial for k-plexes of order $n \geq k+1$ is given by $s_n^k(x) = Z(S_n^{(k+1)}, 1+x)$, and the number of self-complementary k-plexes of order n is thus $\overline{s}_n^k = Z(S_n^{(k+1)}; 0, 2, 0, 2, \ldots)$.* □

7.56. There are several things to note about the number of self-complementary k-plexes. First, there are no self-complementary k-plexes unless $\binom{n}{k+1}$ is even. Secondly, since $S_n^{(r)}$ and $S_n^{(n-r)}$ are identical permutation groups, we have $Z(S_n^{(r)}) = Z(S_n^{(n-r)})$ and, putting $r = k+1$, $\overline{s}_n^k = \overline{s}_n^{n-k-2}$, $1 \leq k \leq n-3$.

It is quickly seen that the unique n-simplex of order n is not self-complementary, while there are $n/2$ self-complementary $(n-2)$-simplexes whenever n is even.

We thus only need to know $\overline{s}_n^k$ for $1 \leq k \leq (n-3)/2$. Palmer tabulated these numbers for $n \leq 9$ (Table 7.9). He also noted that $\overline{s}_8^2$ coincides with $\overline{d}_8$, the number of self-complementary digraphs on 8 vertices, because the polynomials $Z(S_8^{(3)}; 0, x_2, 0, x_4, \ldots)$ and $Z(S_8^{[2]}; 0, x_2, 0, x_4, \ldots)$ are equal; but that this will not happen again because $\binom{n}{k+1} = n(n-1)$ only when $n = 8$ and $k = 2$ or 4.

k / n	sc-graphs 1	2	3
1	1	1	1
2	0	1	1
3	0	0	1
4	1	1	0
5	2	2	0
6	0	40	0
7	0	0	0
8	10	703,760	128
9	36	131,328	16,384

Table 7.9: Self-complementary k-plexes

7.57. The number of k-plexes of order n is given asymptotically by

$$s_n^k = \frac{2^{\binom{n}{k+1}}}{n!}\left(1 + O\left(n^2/2^n\right)\right).$$

In the self-complementary case, Palmer gave an approximation for $k = 2$. The number $\overline{s}_n^2$ is 0 only when $\binom{n}{3}$ is odd, that is whenever $n \equiv 3 \pmod 4$. For an even number of points we have

$$\overline{s}_{2n}^2 \sim \frac{2^{n(n-2)(2n+1)/3}}{n!}$$

and for $n \equiv 1 \pmod 4$ we have

$$\overline{s}_{4n+1}^2 \sim \frac{2^{n(16n^2-9/6}}{2n!}$$

Self-complementary orbits and functions

7.58. Any permutation group Γ acting on a finite object set X partitions X into orbits in the usual way. It also induces a partition of $X^{(2)}$, the collection of all 2-subsets of X, and in general of $X^{(k)}$, where $1 \leq k \leq |X|$. The subsets of $X^{(k)}$ obtained in this way are called the k-orbits of Γ, so that 1-orbits are the usual orbits of Γ. A generalized orbit is a k-orbit for some k. A self-complementary k-orbit S is one in which, for every k-subset K of S, $X - K$ is also in S. In particular this implies that $k = |K| = |X|/2$, so that sc-orbits cannot occur for permutation groups of odd degree.

If we take X to be the set of all pairs of distinct vertices, and Γ to be the pair group $S_n^{(2)}$, then the k-orbits are just isomorphism classes of graphs with n vertices and k edges, and self-complementary orbits are just the usual self-complementary graphs.

This is a very comprehensive concept because most self-complementary structures can be described as sc-orbits of an appropriate permutation group, but it does not cover, say, self-dual nets or colour cyclic factorisations. De Bruijn [51, 52] and Harary and Palmer [179] gave results which implicitly count the number of self-complementary orbits of an arbitrary permutation group. More direct proofs were given later by de Bruijn [53], and Frucht and Harary [131].

Theorem. *The counting polynomial for generalized orbits of a permutation group* Γ *is*

$$f_\Gamma(x) = Z(\Gamma; 1 + x, 1 + x^2, 1 + x^3, \ldots),$$

and the number of self-complementary generalised orbits of Γ *is*

$$\overline{f}_\Gamma = Z(\Gamma; 0, 2, 0, 2, \ldots). \square$$

7.59. Corollary [Frucht and Harary 1974]**.** *For any permutation group* Γ*, if we denote the number of* k*-orbits by* $a_k(\Gamma)$ *we have*

$$\overline{f}_\Gamma = \sum_k (-1)^k a_k(\Gamma).$$

In particular,

$$\begin{aligned}\overline{g}_n &= \sum_k (-1)^k g_{n,k} \\ \overline{d}_n &= \sum_k (-1)^k d_{n,k}.\end{aligned}$$

Proof: We can see that $\overline{f}_\Gamma = f_\Gamma(-1)$, and that $f_\Gamma(x) = \sum_k a_k x^k$, from which the result then follows immediately. □

7.60. A (two-coloured) necklace is a signed circuit. We omit the description "two-coloured", since all the necklaces we consider here will be of this type. If we allow only rotational automorphisms, then the number of necklaces invariant under change of sign (*rotationally self-complementary necklaces*) is given by $s(C_{2n})$ and we have

$$Z(C_{2n}) = \frac{1}{2n}\sum_{k|2n} \phi(k) x_k^{2n/k} \Rightarrow s(C_{2n}) = \frac{1}{2n}\sum_{d|n} \phi(2d) 2^{n/d}$$

which reduces, for $n = p$, an odd prime, to

$$s(C_{2p}) = 1 + \frac{2^{p-1} - 1}{p}.$$

A much more general and detailed treatment of rotationally sc-necklaces, and their links to self-reciprocal polynomials, can be found in [254].

If we also allow reflections as automorphisms, then the number of self-complementary necklaces is just the number of self-complementary orbits of the dihedral group, $s(D_{2n})$:

$$Z(D_{2n}) = \frac{1}{2}Z(C_{2n}) + \frac{1}{4}(x_2^n + x_1^2 x_2 n - 1) \Rightarrow s(D_{2n}) = \frac{1}{2}s(C_{2n}) + 2^{n-2}.$$

This formula implies that $s(C_{2n})$ is always an even number. Note that the number of necklaces rotationally equivalent both to their complement and also to their reflection (*rotationally self-complementary achiral necklaces*) found in [283] cannot be deduced with these methods, as it involves two self-dualities.

The symmetric group S_{2n} and the alternating group A_{2n} have just one self-complementary generalised orbit each, while for the identity group E_{2n} we have

$$Z(E_{2n}) = x_1^{2n} \Rightarrow s(E_{2n}) = 0.$$

7.61. Wille [386] considered an even more general concept by allowing multiple copies of each element in the object set. Let Γ be a permutation group acting on the set X, and consider a function $f : X \to \{0, 1, 2, \ldots, r\}$. The complement of f is defined by $\overline{f}(x) = r - f(x)$, and a self-complementary function is one for which there exists $\gamma \in \Gamma$ such that $f(\gamma(x)) = \overline{f}(x) \forall x \in X$. If we take $r = 1$ we get sc-orbits. If we take D to be the set of all pairs of distinct vertices from $\{1, 2, \ldots, n\}$, and Γ to be $S_n^{(2)}$, then a sc-function is just a self-complementary r-multigraph (7.9). If, instead, we take D to be the set of all ordered pairs, not necessarily distinct, we get the number of self-complementary relations with r variables on n vertices (7.29).

Theorem. *The number of self-complementary functions*

$$f : X \to \{0, 1, 2, \ldots, r\},$$

where X is the object set of a permutation group Γ, is

$$\overline{f}_\Gamma^r = Z(\Gamma; a, r+1, a, r+1, \ldots) \text{ with } a = \begin{cases} 0, & \text{if } r \text{ is odd,} \\ 1, & \text{if } r \text{ is even.} \end{cases}$$

7.62. Finally we note that Harary announced the number of what he called self-complementary configurations in [177], and these were also treated by Palmer and Robinson [281]. The latter paper also contains the number of sc-boolean functions, which is also treated in [110, 196, 197, 272, 282].

7.63. To the reader: If it is true that with each equation the number of potential readers drops by half, then the mere fact that you have arrived here would prove that we are far outnumbered by extraterrestrials.

Open problems

7.64. There are a number of counting problems which still remain unsolved today, in particular

A. labelled self-complementary graphs [Harary and Palmer 1973, p.217]

B. labelled self-complementary digraphs (*ibid.*)

C. labelled self-converse digraphs (*ibid.*)

D. potentially self-complementary degree sequences [Rao 1979a]

E. self-dual digraphs — those which are both self-converse and self-complementary [Harary 1967]

F. doubly self-dual nets [Harary, Palmer, Robinson and Schwenk 1977]

G. strongly regular sc-graphs

There are also a number of unexplained correspondences. In particular no one has yet found natural bijections between

H. self-complementary graphs on $4n$ vertices and self-complementary digraphs on $2n$ vertices;

I. self-complementary relations on $2n$ vertices and either self-complementary symmetric relations on $4n$ vertices or self-complementary graphs on $4n + 1$ vertices;

J. self-complementary two-graphs on $4n + 1$ vertices and either Eulerian sc-graphs on $4n + 1$ vertices, or sc-graphs on $4n$ vertices;

K. circulant self-complementary graphs on $2p - 1$ vertices and circulant self-complementary digraphs on p vertices, where p and $2p-1$ are both odd primes.

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